2. *An Extension of Beurling's Theorem. I*

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Let R be a Riemann surface with positive boundary and let $\{R_n\}$ $(n=0, 1, 2, \cdots)$ be its exhaustion with compact relative boundary ∂R_n such that $\partial R_n \cap \partial R_{n+1}=0$. Let $N(z, p)$ be a positive harmonic function in $R-R_0-p: p\in R-R_0$ such that $N(z, p)=0$ on ∂R_0, $N(z, p)$ has a logarithmic singularity at p and $N(z, p)$ has minimal Dirichlet integral over $R-R_0$, where Dirichlet integral is taken with respect to $N(z, p)+\log|z-p|$ in a neighbourhood of p. We call such $N(z, p)$ an N-Green's function with pole at p. Consider now a sequence of points $\{p_i\}$ of $R-R_0$ having no points of accumulation in $R-R_0+\partial R_0$. Since the functions $N(z, p_i)$ $(i=1, 2, \cdots)$ forms, from some i on, a bounded sequence of harmonic functions—thus a normal family. A sequence of these functions, therefore is convergent in every compact part of $R-R_0$ to a positive harmonic function. A sequence $\{p_i\}$ of $R-R_0$ having no point of accumulation in $R-R_0+\partial R_0$, for which the corresponding $\{N(z, p_i)\}$ have the property just mentioned, that is, $\{N(z, p_i)\}$ converges to a harmonic function—will be called fundamental. If two fundamental sequences determine the same limit function $N(z, p)$, we say that they are equivalent. Two fundamental sequences equivalent to a given one determine an ideal boundary point of R. The set of all the ideal boundary points of R will be denoted by B and the set $R-R_0+B$ by $\bar{R}-R_0$. The domain of definition of $N(z, p)$ may now be extended by writing $N(z, p)=\lim_i N(z, p_i)$ $(z\in R-R_0,\ p\in \bar{R}-R_0)$, where $\{p_i\}$ is any fundamental sequence determining p. The function $N(z, p)$ is characteristic of the point p of their corresponding $N(z, p)$ as a function of z. The distance $\delta(p_1, p_2)$ of two points p_1 and p_2 in $\bar{R}-R_0$ is defined as

$$\delta(p_1, p_2)=\sup_{z\in R_1}\left|\frac{N(z, p_1)}{1+N(z, p_1)}-\frac{N(z, p_2)}{1+N(z, p_2)}\right|.$$

The topology (N-Martin's topology) [1] is induced by this metric.

Let $U(z)$ be a positive superharmonic function in $R-R_0$ such that $D(\min(M, U(z)))<\infty$ for every M and $U(z)=0$ on ∂R_0. Let G be a domain [2] in $R-R_0$ and let ${}_GU^M(z)$ be a superharmonic function in $R-R_0$ such that ${}_GU^M(z)=\min(M, U(z))$ on $G+\partial R_0$ and ${}_GU^M(z)$ has minimal Dirichlet integral. Put ${}_GU(z)=\lim_{M=\infty} {}_GU^M(z)$. If for any domain G, ${}_GU(z)\leqq U(z)$, $U(z)$ is called a full-superharmonic function

[3] in $R-R_0$. We see $N(z, p)$ is full-superharmonic in $R-R_0$. To every point $p\in \bar{R}-R_0$ an N-Green's function corresponds. B consists of two parts, B_1^N, the set of N-minimal point and the set B_0^N, the set of non N-minimal points, where B_0^N is an F_σ set of capacity zero. It is known that $N(z, p): p\in R-R_0+B_1^N$ has many properties as the function $-\log|z-p|$ in the z-plane, for instance, $N(z, p)=\lim_{M=M^*} {}_{V_M(p)}N(z, p)$, where $V_M(p)=E[z\in R-R_0 : N(z, p)>M]$ and $M^*=\sup_{Z\in R} N(z, p)$. Let $G_1\supset G_2$ be domains. Let $\omega(G_2, z, G_1)$ be a continuous function in G_1 such that $\omega(G_2, z, G_1)=0$ on ∂G_1, $=1$ on G_2, and $\omega(G_2, z, G_1)$ is harmonic in G_1-G_2 and has M.D.I. (minimal Dirichlet integral)$<\infty$. We call $\omega(G_2, z, G_1)$ C.P. (Capacitary potential) [4] of G_2 relative to G_1.

Let $\{G_n\}(n=0, 1, 2, \cdots)$ be a decreasing sequence of domains in $R-R_0$. Let $\omega_n(z)=\omega(G_n, z, G_0)$, where $\omega_n(z)$ has M.D.I. $<\infty$ for $n\geq n_0$ and n_0 is a certain number. Then $\omega_n(z)$ converges in mean (we denote it by $\Rightarrow$) to a harmonic function in $G_0-(\lim_n G_n)$ denoted by $\omega(\{G_n\}, z, G)$ as $n\to\infty$. If $\{G_n\}$ tends to the boundary, we call $\omega(\{G_n\}, z, G)$ the C.P. of the ideal boundary determined by $\{G_n\}$. If $G_0=R-R_0$, we simply denote by $\omega(\{G_n\}, z)$. It is known if $\omega(\{G_n\}, z, G_0)>0$, $\sup_{z\in R}\omega(\{G_n\}, z, G_0)=1$ [5].

Let $p\in B_1^N$. Then to cases occur (1) $\sup_{z\in R} N(z, p)=\infty$ (this is equivalent to $\omega(p, z)=\lim_{n=\infty}\ \omega(v_n(p), z)=0$) and (2) $\sup_{z\in R} N(z, p)<\infty$ (this is equivalent to $\omega(p, z)>0$), where $v_n(p)=E\left[z\in \bar{R}: \delta(z, p)<\frac{1}{n}\right]$. We denote by B_S^N the set of $p\in B$ such that $\omega(p, z)>0$. Then $B_S^N\subset B_1^N$.

Contact set $\varDelta(p)$ of $p\in B_1^N$. Suppose $p\in R-R_0+B_1^N$. Then $N(z, p)=\lim_{n=\infty} {}_{v_n(p)}N(z, p)=N(z, p)$. Let $\varDelta(p)$ be a closed set in R. If $\overline{\lim_{n=\infty}}\ {}_{\varDelta(p)\cap v_n(p)}N(z, p)(=\lim_{n=\infty} {}_{\varDelta(p)\cap v_n(p)}N(z, p))>0$, we call $\varDelta(p)$ a contact set of p. Clearly $\lim_{n=\infty} {}_{\varDelta(p)\cap v_n(p)}N(z, p)$ has mass only at p, whence $\lim_{n=\infty} {}_{\varDelta(p)\cap v_n(p)}N(z, p)=\alpha N(z, p)$: $1\geqq\alpha\geqq 0$. If $N(z, p)-{}_{CG}N(z, p)>0$ (this is equivalent to that CG is thin at p), we denote by $G\overset{N}{\ni} p$. It is well known $v_n(p)\overset{N}{\ni} p$ and $V_M(p)\overset{N}{\ni} p$ [6] for $M<M^*=\sup_{z\in R} N(z, p)$.

Lemma 1.1). *Suppose* $G\overset{N}{\ni} p$, *then* ${}_{CG\cap P}N(z, p)=\lim_{n=\infty} {}_{CG\cap v_n(p)}N(z, p))$ $=0$.

2). *Let* $\varDelta(p)$ *be a contact set of* p. *Then* $(R-\varDelta(p))\overset{N}{\not\ni} p$. *This means that* $\varDelta(p)$ *is not contained in any thin set at* p.

3). *Let* $\varDelta(p)$ *be a contact set and suppose* $G\overset{N}{\ni} p$. *Then* $\varDelta(p)\cap G$ *is also a contact set.*

Proof of 1). *Case* 1. $p\in B_1^N-B_S^N$, i.e. $\omega(p, z)=0$. Suppose $G\overset{N}{\ni} p$

and assume ${}_p({}_{CG}N(z,p))>0$. Then ${}_p({}_{CG}N(z,p))$ has mass only at p, whence ${}_p({}_{CG}N(z,p))=\alpha N(z,p)>0$, ${}_{CG}N(z,p)-{}_p({}_{CG}N(z,p))=U(z)$ is also full-superharmonic [7] and ${}_{CG}U(z)\leqq U(z)$. Now ${}_{CG}N(z,p)=\alpha N(z,p)+U(z)$. Clearly ${}_{CG}({}_{CG}Nz,p))={}_{CG}N(z,p)$. We have

$$ {}_{CG}({}_{CG}N(z,p))=\alpha_{CG}N(z,p)+{}_{CG}U(z)=\alpha N(z,p)+U(z)={}_{CG}N(z,\mathrm{p}). $$

On the other hand, ${}_{CG}N(z,p)\leqq N(z,p)$ and ${}_{CG}U(z)\leqq U(z)$, whence we have $\alpha N(z,p)=\alpha_{CG}N(z,p)$. This contradicts $G\overset{N}{\ni}p$. Hence ${}_p({}_{CG}N(z,p))=0$. Assume $0<{}_{p\cap CG}N(z,p)=\lim_{n=\infty}{}_{v_n(p)\cap CG}N(z,p)$. Then ${}_{p\cap CG}N(z,p)=\beta N(z,p)+U'(z):\beta>0$, where $U'(z)$ is full-superharmonic. Whence ${}_{CG}N(z,p)\geqq{}_{p\cap CG}N(z,p)\geqq\beta N(z,p)$ and we have ${}_p({}_{CG}N(z,p))\geqq\beta N(z,p)>0$. This contradicts ${}_p({}_{CG}N(z,p))=0$. Thus ${}_{p\cap CG}N(z,p)=0$.

Case 2. $p\in B_S^N\subset B_1^N$. In this case $\omega(p,z)>0$, $\sup_{z\in R}N(z,p)<\infty$ and we can use $\omega(p,z)$ instead of $N(z,p)$. Assume ${}_{p\cap CG}\omega(p,z)=\lim_{n=\infty}{}_{v_n(p)\cap CG}\omega(p,z)>0$. For any $\varepsilon>0$ we can find a number n_0 such that $1\geqq\omega(p,z)\geqq1-\varepsilon$ in $v_n(p)$ [8] for $n\geqq n_0$. We have

$$ \omega(CG\cap v_n(p),z)\geqq{}_{CG\cap v_n(p)}\omega(p,z)\geqq(1-\varepsilon)\omega(CG\cap v_n(p),z). $$

Let $n\to\infty$ and then $\varepsilon\to0$. Then

$$ ({}_{CG}\omega(p,z)\geqq)_{CG\cap p}\omega(p,z)=\omega(CG\cap p,z)>0. $$

Now $\omega(CG\cap p,z)>0$ implies $\sup_{z\in R}\omega(CG\cap p,z)=1$ and $\omega(CG\cap p,z)$ has mass only at p, whence $\omega(CG\cap p,z)=\omega(p,z)$. Hence ${}_{CG}\omega(p,z)=\omega(p,z)$. This contradicts $G\overset{N}{\ni}p$. Hence ${}_{CG\cap p}\omega(p,z)=0$ and ${}_{CG\cap p}N(z,p)=0$.

Proof of 2). By 1) we have $\lim_{n=\infty}{}_{v_n(p)\cap CG}N(z,p)=0$. Hence CG is not a contact set.

Proof of 3). Also by 1)

$$ 0<\lim_{n=\infty}{}_{\Delta(p)\cap v_n(p)}N(z,p)\leqq\lim_{n=\infty}{}_{\Delta(p)\cap v_n(p)\cap CG}N(z,p) $$
$$ +\lim_{n=\infty}{}_{\Delta(p)\cap v_n(p)\cap G}N(z,p)=\lim_{n=\infty}{}_{\Delta(p)\cap v_n(p)\cap G}N(z,p). $$

Hence $G\cap\Delta(p)$ is a contact set of p. *A sufficient condition for a set Δ to be a contact set of $p\in B_1^N$.* By Theorem 6 of the previous paper (C) [9] we have the following

Lemma 2). *If there exists a sequence $M_1<M_2,\cdots<M^*=\sup_{z\in R}N(z,p)$ such that*

$$ \overline{\lim_{M_i\to M^*}}\int_{\partial V_{M_i}(p)\cap\Delta}\frac{\partial}{\partial n}N(z,p)ds>0. $$

Then Δ is a contact set of p.

In the following we consider contact sets when a Riemann surface is very simple. Let R be a unit circle $|z-1|<1$. We suppose N-Martin's topology is defined in $R-R_0$. Then we have $B_0^N=0$ and every point $e^{i\theta}$ is an N-minimal boundary point.

Lemma 3.1). *Let $F=\sum_{n=0}^{\infty}F_n$ be a closed set in $|z-1|<1$ such* *that* $\{F_n\}$ *tends to* $z=0$ *as* $n\to\infty$ *and* F_n *is a connected component. Let* F_n^p *be the circular projection of* F_n *on the positive real axis such that* $F_n^p=E[z:r'_n\leqq Re\,z\leqq r_n]$, $r_n=\max\limits_{z\in F_n}|z|$ *and* $r'_n=\min\limits_{z\in F_n}|z|$. *Put* $\delta_n=r_n-r'_n$. *Then*

Condition (A). *If* $\overline{\lim\limits_{n=\infty}}\dfrac{\log r_n}{\log\delta_n}>0$, *then* F *is a contact set of* $z=0$.

Condition (A) *means there exists a const.* $M<\infty$ *and infinitely many numbers* n_i *such that* $\delta_{ni}>r_{ni}^M$.

We can suppose without loss of generality $R_0=E\left[z:|z-1|<\dfrac{1}{2}\right]$. Let $\hat{R}-\hat{R}_0$ and $\hat{F}$ be symmetric images of $R-R_0$ and of F with respect to the circle $C:|z-1|=1$ respectively. Let $\tilde{R}-\tilde{R}_0=R-R_0+C+\hat{R}-\hat{R}_0$. Then $\tilde{R}-\tilde{R}_0$ is a ring domain $\dfrac{1}{2}<|z-1|<2$. Let $N(z,0)$ be the N-Green's function of $R-R_0$ corresponding to $z=0$. Then $N(z,0)=2\,G(z,0)=-2\log|z|+V(z)$, where $G(z,0)$ is the Green's function of $\tilde{R}-\tilde{R}_0$ with pole at $z=0$ and $V(z)$ is a harmonic function in a neighbourhood in $\tilde{R}-\tilde{R}_0$ of $z=0$. Let $\{v_n(0)\}$ be a system of neighbourhood of the boundary point $z=0$ with respect to N-Martin's topology and let $v_n^E(0)=E\left[z\in\tilde{R}-\tilde{R}_0:|z|<\dfrac{1}{n}\right]$. Then systems $\{v_n(0)+\hat{v}_n(0)\}$ and $\{v_n^E(0)\}$ are equivalent, where $\hat{v}_n(0)$ is the symmetric image of $v_n(0)$ with respect to C. *We show* $\lim\limits_{n=\infty}{}_{v_n(0)\cap F}N(z,0)>0$ under the condition (A). Now

$$ {}_{v_n(0)\cap F}N(z,0)=2\ {}_{(v_n(0)+\hat{v}_n(0))\cap(F+\hat{F})}G(z,0), $$

where ${}_{(v_n(0)+\hat{v}_n(0))\cap(F+\hat{F})}G(z,0)$ is the lower envelope of positive superharmonic functions in $\tilde{R}-\tilde{R}_0$ larger than $G(z,0)$ on

$$ (v_n(0)+\hat{v}_n(0))\cap(F+\hat{F}). $$

Let ${}_{(v_n(0)+\hat{v}_n(0))\cap(F+\hat{F})}U(z)$ and ${}_{v_n(p)\cap F}U^*(z)$ be lower envelopes of positive superharmonic functions in $\Gamma:|z|<1$ larger than $-\log|z|$ on $(v_n(0)+\hat{v}_n(0))\cap(F+\hat{F})$ and larger than $-\log|z|$ on $v_n(0)\cap F$ respectively. Then since $V(z)$ is bounded in a neighbourhood of $z=0$, we have

$$ \begin{aligned} \lim_{n=\infty}{}_{v_n(0)\cap F}N(z,0) &= \lim_{n=\infty}2\ {}_{(v_n(p)+\hat{v}_n(p))\cap(F+\hat{F})}G(z,0) \\ &\geqq \lim_{n=\infty}{}_{(v_n(p)+\hat{v}_n(p))\cap(F+\hat{F})}U^*(z)\geqq\overline{\lim_{n=\infty}}\ {}_{v_n(0)\cap F}U^*(z) \\ &= \overline{\lim_{n=\infty}}\ {}_{v_n^E(p)\cap F}U(z)\geqq\overline{\lim_{n=\infty}}\ {}_{F_n}U^*(z)\geqq\overline{\lim_{n=\infty}}\ U_n(z), \end{aligned} $$

where ${}_{F_n}U^*(z)$ and $U_n(z)$ are lower envelopes of positive superharmonic function in $|z|<1$ larger than $-\log|z|$ on F_n and larger than $-\log r_n$ on F_n respectively (because $-\log|z|\geqq-\log r_n$ on F_n).

We estimate the module of a ring domain $(\Gamma-F_n)$. Let p and q be two points such that $p=r'_ne^{i\theta}$, $q=r_ne^{i\varphi}$, where $r_n=\max\limits_{z\in F_n}|z|$ and

$r'_n = \min_{z \in F} |z|$. Then F_n contains at least a curve γ connecting p with q. Then by $F_n \supset \gamma$, module of $(\Gamma - F_n)$ is smaller than that of $(\Gamma - \gamma)$. Map $\Gamma - \gamma$ by

$$w = \frac{1 - r'_n e^{-i\theta} z}{z - r'_n e^{i\theta}}.$$

Then $\Gamma - \gamma$ is mapped onto a ring whose boundary consists of $|w| = 1$ and a curve γ_w connecting $w = \infty$ with $w = \frac{1 - r_n r'_n e^{-i\theta + i\varphi}}{r_n e^{i\varphi} - r'_n e^{-i\theta}}$. Now $\left| \frac{1 - r_n r'_n e^{-i\theta + i\varphi}}{r_n e^{i\varphi} - r'_n e^{i\theta}} \right| \leqq \frac{2}{r_n - r'_n}$. Let Ω be a Koebe's extremal ring domain such that $\partial\Omega$ consists of $|w| = 1$ and a half straight line on the real axis connecting $w = \infty$ with $w = \frac{2}{r_n - r'_n} > 1$. Then the module of $(\Gamma - \gamma)$ is smaller than that of $\Omega \leqq \log \frac{4 \times 2}{r_n - r'_n}$. $U_n(z)$ is a harmonic function in $\Gamma - \gamma$ such that $U_n(z) = 0$ on $\partial\Gamma$ and $U_n(z) = -\log r_n$ on γ, whence

$$\int_{\partial\Gamma} \frac{\partial}{\partial n} U_n(z) ds \geqq \frac{2\pi(-\log r_n)}{\text{mod. of } (\Gamma - \gamma)} \geqq \frac{-2\pi \log r_n}{\log \frac{8}{r_n - r'_n}} \geqq \frac{2\pi \log r_n}{\log \delta_n} > 0.$$

Hence $\lim_{n=\infty} {}_{v_n(p) \cap F} N(z, p) \geqq \overline{\lim_n} U_n(z) > 0$ and F is a contact set of $z = 0$. As an application of Lemma 3), 1) we have at once the following

Lemma 3. 2). *Let R be a Riemann surface such that $|z| < 1$. Let γ be a curve terminating at $e^{i\theta}$. Then γ is a contact set of $e^{i\theta}$.*

Since $N(z, 0) + 2\log|z|$ is harmonic in a neighbourhood of $z = 0$ in $\tilde{R} - \tilde{R}_0$ and by Lemma 2 we have at once

Lemma 3. 3). *Let R be the same Riemann surface as Lemma 3).1. Let $F = \sum_n F_n$ be a closed set in R such that $\{F_n\}$ tends to $z = 0$ as $n \to \infty$ and every F_n contains a circular arc: $E[z : |z| = r_n, \theta_n \leqq \arg z \leqq \theta_n + \delta_n]$. Then*

Condition (B). *If $\overline{\lim_{n=\infty}} \delta_n > 0$, F is a contact set of $z = 0$.*

Let R be $|z - 1| < 1$. Then we see F is thin at $z = 0$ (this is equivalent to $R - F \overset{N}{\ni}$ the point $z = 0$), if and only if $z = 0$ is regular for the Dirichlet problem in a domain $\Gamma - F - \hat{F}$, where $\Gamma = E\left[z : \frac{1}{2} < |z - 1| < 2\right]$ and $\hat{F}$ is the symmetric image of F with respect to $|z - 1| = 1$. Hence by Lemma 2 we have

Theorem 1. *Conditions* (A) *and* (B) *are sufficient conditions for $z = 0$ to be regular for the Dirichlet problem in $\Gamma - F - \hat{F}$.*

Let $G_1 \supset G_2$ be two domains. If there exists a C_1-function $U(z)$ in G_1 [10] such that $U(z) = 0$ on ∂G_1, $U(z) = 1$ on G_2 and the Dirichlet integral $D(U(z)) < \infty$, we say CG_1 and G_2 are Dirichlet-disjoint. Let

$\omega(\{G_n\}, z, G_0)$ be $C.P.$ of the boundary determined by $\{G_n\}$. Then we proved

Lemma 4. 1). [11] *Let* $\omega(\{G_n\}, z, G_0)>0$. *Then there exists a level curve* C_r *of* $\omega(\{G_n\}, z, G_0)$ *such that*

$$\int_{C_r}\frac{\partial}{\partial n}\omega(\{G_n\}, z, G_0)ds = D(\omega(\{G_n\}, z, G_0))$$

for almost $r: 0\leqq r\leqq 1$.

2). [12] *If* G_{n+i} *and* CG_n *are Dirichlet-disjoint, for any* G_n

$$\int_{C_r\cap CG_n}\frac{\partial}{\partial n}\omega(\{G_n\}, z, G_0)ds \downarrow 0 \text{ as } r\uparrow 1.$$

3). If CG_0 and G_{n0} (n_0 is a certain number) are Dirichlet-disjoint, we have by the Dirichlet principle and by maximum principle

$$\omega(\{G_n\}, z, G_0)>0 \quad \text{if and only if} \quad \omega(\{G_n\}, z)(=\omega(\{G_n\}, z, R-R_0))>0.$$

References

[1] Z. Kuramochi: Potentials on Riemann surfaces. Jour. Fac. Sci. Hokkaido Univ., XVI (1962).

[2] In the present articles we suppose ∂G consist of enumerably infinite number of components clustering nowhere in R.

[3] See [1] But in 1) full-superharmoic functions called superharmonic functions.

[4] See [1].

[5] See [1].

[6] Z. Kuramochi: Singular points of Riemann surfaces. Jour. Fac. Sci. Hokkaido Univ., XVI (1962).

[7] See Theorem 6 of [1].

[8] See [6].

[9] C means the paper "Correspondence of Boundaries of Riemann surfaces. Jour. Fac. Sci. Hokkaido Univ., XVII (1963).

[10] If $g(z)$ is continuous and partially defferentialble almost everywhere, $g(z)$ is called a C_1-function.

[11] See Lemma 1 of C (See [9]).

[12] See [11].