9. *Ackermann's Model and Recursive Predicates*

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Let N be the set of all non-negative integers. Define a binary predicate $\in$ on N by

$$a \in b. \equiv. [b/2^a] \text{ is odd},$$

where $[x]$ means the greatest integer contained in x. (For the recursive definition of $[x/y]$, see Kleene [1], p. 223). Then the structure $\langle N, \in\rangle$, which is called Ackermann's model, satisfies all the axioms of ZF except the axiom of infinity.

A predicate $P(a_1, \cdots, a_n)$ on N is called bounded, if there exists a restricted formula $A(x_1, \cdots, x_n)$ in the sence of [2] such that $P(a_1, \cdots, a_n)$ holds if and only if $A(a_1, \cdots, a_n)$ is true in $\langle N, \in\rangle$. Then our main theorem can be stated as follows:

Theorem. *A predicate $R(a_1, \cdots, a_n)$ is general recursive if and only if there exists bounded predicates $P(a, a_1, \cdots, a_n)$ and $Q(a, a_1, \cdots, a_n)$ such that*

(1) $\quad R(a_1, \cdots, a_n) \equiv \exists x P(x, a_1, \cdots, a_n) \equiv \forall x Q(x, a_1, \cdots, a_n)$

for all $a_1, \cdots, a_n \in N$.

Proof. First suppose that there exist P and Q satisfying (1). Since $\in$ is primitive recursive, we can easily show that every bounded predicate is primitive recursive. Hence, by the theorem VI(b) of [1], R is general recursive. Before proving the converse, we prove several lemmata. We temporarily call a predicate R for which there can be found bounded predicates P and Q satisfying (1) as a Δ-predicate.

Lemma 1. *$a<b$ is a Δ-predicate.*

Proof. Let $A(p, z). \equiv.$ $\mathrm{Comp}(z) \wedge p \subseteq z \times z \wedge \forall x \forall y (\langle xz\rangle \in p \equiv x \in z \wedge y \in z \wedge \exists u (u \in y \wedge u \notin x \wedge \forall v (\langle uv\rangle \in p \supset (v \in x \equiv v \in y))))$, where $z \times z$ means direct product. Then $A(p, z)$ has the following properties:

1° $A(p, z)$ is bounded.

2° If $A(p, z)$, then we have

$$\forall i \forall j (\langle ij\rangle \in p \equiv i \in z \wedge j \in z \wedge i < j).$$

3° $\forall a \forall b \exists p \exists z (a \in z \wedge b \in z \wedge A(p, z))$.

1° and 3° are easily proved. 2° is proved by the induction on $\max(i, j)$. Therefore

$$a<b \equiv \overset{\forall}{\exists} p \overset{\forall}{\exists} z (a \in z \wedge b \in z \wedge A(p, z) \overset{\supset}{\wedge} \langle ab\rangle \in p).$$

This clearly shows $a<b$ is a Δ-predicate.

Lemma 2. *$a'=b$ is a Δ-predicate.*

Let $B(p, z) . \equiv .$ $\operatorname{Comp}(z) \wedge p \subseteq z \times z \wedge \forall x(x \in z \wedge \exists t(t \in x) \supset \exists y(\langle yx \rangle \in p)) \wedge \forall x \forall y(\langle xy \rangle \in p . \equiv . x \in z \wedge y \in z \wedge \exists u(u \in y \wedge u \notin x \wedge \forall t(t \in z \wedge t < u \supset t \in x \wedge t \notin y) \wedge \forall t(t \in x \wedge u < t \supset (t \in x \equiv t \in y))))$.
Then

1° $B(p, z)$ is a Δ-predicate.

2° If $B(p, z)$. then we have

$$\forall i \forall j(\langle ij \rangle \in p \equiv i \in z \wedge j \in z \wedge i' = j)$$

3° $\forall a \forall b \exists p \exists z(a \in z \wedge b \in z \wedge B(p, z))$.

2° is proved also by the induction on j. Therefore

$$a' = b \equiv \overset{\forall}{\exists} p \overset{\forall}{\exists} z(a \in z \wedge b \in z \wedge B(p, z) \overset{\supset}{\wedge} \langle ab \rangle \in p).$$

Lemma 3. *If $\varphi(a_1, \cdots, a_n)$ is primitive recursive, then $\varphi(a_1, \cdots, a_n) = b$ is a Δ-predicate.*

Proof. *Case* I. $\varphi(a) = a'$. Use Lemma 2.

Case II. $\varphi(a_1, \cdots, a_n) = q$.

$\varphi(a_1, \cdots, a_n) = b . \equiv . b = q . \equiv . \forall t(t \in b \supset t \in q) \wedge \forall t(t \in q \supset t \in b)$.

Right most formula is bounded and hence a Δ-predicate.

Case III. $\varphi(a_1, \cdots, a_n) = a_i$.

$$\varphi(a_1, \cdots, a_n) = b . \equiv . b = a_i.$$

Case IV. $\varphi(a_1, \cdots, a_n) = \psi(\chi_1(a_1, \cdots, a_n), \cdots, \chi_m(a_1, \cdots, a_n))$

$\varphi(a_1, \cdots, a_n) = b . \equiv . \overset{\forall}{\exists} z_1 \cdots \overset{\forall}{\exists} z_m(\chi_1(a_1, \cdots, a_n) = z_1 \wedge \cdots \chi_m(a_1, \cdots, a_n) = z_m \overset{\supset}{\wedge} \psi(z_1, \cdots, z_m) = b)$

Case Va. $\varphi(0) = q, \varphi(a') = \psi(a, \varphi(a))$.

$\varphi(a) = b . \equiv . \overset{\forall}{\exists} p \overset{\forall}{\exists} z \overset{\forall}{\exists} f(B(p, z) \wedge a \in z \wedge \forall i \forall j \forall k(\langle ij \rangle \in f \wedge \langle ik \rangle \in f \supset i = k) \wedge \forall i(i \in z \supset \exists j(\langle ij \rangle \in f \wedge ((i = 0 \wedge j = q) \vee \exists k(k \in z \wedge k' = i \wedge \exists u(\langle ku \rangle \in f \wedge j = \psi(k, u)))))) \overset{\supset}{\wedge} ab \in f)$.

Case Vb. Similar to the case *Va*.

Proof of main theorem. First we assume that $R(a_1, \cdots, a_n)$ is primitive recursive and let $\varphi(a_1, \cdots, a_n)$ be the representing function of it. By the preceding lemma $\varphi(a_1, \cdots, a_n) = 0$ is a Δ-predicate. Hence $R(a_1, \cdots, a_n)$ is a Δ-predicate.

Next let $R(a_1, \cdots, a_n)$ be general recursive. Then there exist primitive recursive predicates $R_1(a, a_1, \cdots, a_n)$ and $R_2(a, a_1 \cdots a_n)$ such that

$$R(a_1, \cdots, a_n) \equiv \exists x R_1(x, a_1, \cdots, a_n) \equiv \forall x R_2(x, a_1, \cdots, a_n).$$

But R_1 and R_2 are Δ-predicates. Hence R is a Δ-predicate. q.e.d.

References

[1] S. C. Kleene: Introduction to Metamathematics. New York and Tronto (Van Nostrand), Amsterdam (North Holland), and Groningen (Noordhoff), (1952).

[2] A. Lévy: A hierarchy of formulas in set theory. Memoirs Amer. Math. Soc., **57**, 76 (1965).