35. Note on the Nuclearity of Some Function Spaces. II

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In this note, by the same method as [3], we shall prove the nuclearity of $Z_\Omega\{M_A\}$, which it is introduced by I. M. Gelfand and G. E. Shilov [2].

Definition. Let A be any index set and we assume that, for each element α of A, $M_\alpha(z)$ is a real valued continuous function defined on a open subset Ω of the complex space C^n and it satisfies the following condition: for each $\alpha \in A$, $M_\alpha(z)$ is positive and

$$\text{if } \alpha \leqq \beta \quad \text{then} \quad M_\alpha(z) \leqq M_\beta(z)\ .$$

We put
$$\|\psi\|_\alpha = \sup_{z \in \Omega} M_\alpha(z)\,|\psi(z)| \qquad (\alpha \in A) \tag{1}$$
where ψ is a element of the set of all entire functions on Ω. Then we denote by $Z_\Omega\{M_A\}$ the set of all the entire functions ψ which satisfies $\|\psi\|_\alpha < \infty$ for all $\alpha \in A$ and the topology of $Z_\Omega\{M_A\}$ be defined by the sequence of norms $\|\psi\|_\alpha (\alpha \in A)$.

We shall prove below that $Z_\Omega\{M_A\}$ is a nuclear space if the following two conditions are satisfied.

(N_1^0) For any element α of A there exists an index $\beta \geqq \alpha$ such that $\dfrac{M_\alpha(z)}{M_\beta(z)}$ is integrable on Ω and if Ω is an unbounded open subset then $\lim\limits_{|z|\to\infty} \dfrac{M_\alpha(z)}{M_\beta(z)} = 0$.

(N_2^0) For any index $\alpha \in A$ there exists an index $\beta \geqq \alpha$ such that, for some positive number γ, if $|w-z| \leqq \gamma$ then
$$\frac{M_\alpha(z)}{M_\beta(w)} \leqq C_\alpha \tag{2}$$
where C_α is a constant number depending on α.

Lemma. *If the condition (N_1^0) holds then the initial topology of the space $Z_\Omega\{M_A\}$ is equivalent to the topology introduced by the sequence of semi-norms*
$$\|\psi\|_{\alpha,K} = \sup_{z \in K} \{M_\alpha(z)\,|\psi(z)|\} \qquad \text{for } \psi \in Z_\Omega\{M_A\}\ , \tag{3}$$
where α be any index in A and K runs all compact subset of Ω.

Proof. Clearly for any $\psi \in Z_\Omega\{M_A\}$
$$\|\psi\|_{\alpha,K} \leqq \|\psi\|_\alpha \tag{4}$$
for all $\alpha \in A$ and compact subset K of Ω.

Next, when Ω is unbounded, for each $\alpha \in A$ and $\psi \in Z_\Omega\{M_A\}$
$$\lim_{|z|\to\infty} M_\alpha(z)\psi(z) = 0\ .$$

Indeed, if this is not true, then for some α there exists a sequence $\{z_m\}$ with $|z_m|\longrightarrow\infty$ such that

$$M_\alpha(z_m)|\psi(z_m)|\geqq C>0.$$

But then by (N_1^0), for any $\varepsilon>0$, there exist a natural number N and an index $\beta\geqq\alpha$ such that

$$\text{if} \quad |z_m|>N \quad \text{then} \quad M_\alpha(z_m)<\varepsilon M_\beta(z_m).$$

Hence, $M_\beta(z_m)|\psi(z_m)|>\dfrac{C}{\varepsilon}$ for $|z_m|>N$

i.e.
$$M_\beta(z_m)|\psi(z_m)|\longrightarrow\infty \qquad \text{as } m\longrightarrow+\infty$$

which is in contradiction with $\|\psi\|_\beta<\infty$. Therefore, for any $\alpha\in A$ there exists a compact subset K such that

$$\|\psi\|_\alpha=\|\psi\|_{\alpha,K} \tag{5}$$

for all $\psi\in Z_\Omega\{M_A\}$. When Ω is bounded, it is evident by continuity of $M_\alpha(Z)$ that a compact set K satisfying (5) exists. From (4) and (5), the proof is completed.

Theorem. *If the space $Z_\Omega\{M_A\}$ satisfies the conditions (N_1^0) and (N_2^0), then it is a nuclear space.*

Proof. For any $\alpha\in A$, by (N_1^0), there exists an index β such that $\dfrac{M_\alpha(z)}{M_\beta(z)}$ is integrable on Ω. Hence if $\psi\in Z_\Omega\{M_A\}$ then

$$M_\alpha(z)\,|\psi(z)|=\frac{M_\alpha(z)}{M_\beta(z)}M_\beta(z)\,|\psi(z)|\leqq\frac{M_\alpha(z)}{M_\beta(z)}\sup_{z\in\Omega}M_\beta(z)\,|\psi(z)|.$$

Therefore

$$\int_\Omega M_\alpha(z)\,|\psi(z)|\,dz\leqq\Big(\sup_{z\in\Omega}M_\beta(z)\,|\psi(z)|\Big)\int_\Omega\frac{M_\alpha(z)}{M_\beta(z)}dz<\infty.$$

Next, for any compact subset $K\equiv\{w:|w-z_0|\leqq\rho\}$ of Ω with $0<\rho<\dfrac{r}{2}$ there exists ε with $0<\varepsilon<r-2\rho$ such that

$$H\equiv\{w:|w-z_0|\leqq\rho+\varepsilon\}\subset\Omega.$$

Then for every $\psi\in Z_\Omega\{M_A\}$

$$\psi(z)=\frac{1}{2\pi i}\int_{|w-z_0|=\rho+\varepsilon}\frac{f(w)}{w-z}dw \qquad \text{for } z\in K.$$

But since $\varepsilon\leqq|w-z|\leqq|w-z_0|+|z_0-z|\leqq2\rho+\varepsilon\leqq r$ therefore

$$|\psi(z)|\leqq\frac{(\rho+\varepsilon)}{2\pi\varepsilon}\int_0^{2\pi}|\psi(w)|\,d\theta$$

where $w=z_0+(\rho+\varepsilon)e^{i\theta}$ henceforth by (N_2^0)

$$M_\alpha(z)\,|\psi(z)|\leqq\frac{C_\alpha(\rho+\varepsilon)}{2\pi\varepsilon}\int_0^{2\pi}M_\beta(w)\,|\psi(w)|\,d\theta$$

i.e.
$$\|\psi\|_{\alpha,K}\leqq\frac{C_\alpha(\rho+\varepsilon)}{2\pi\varepsilon}\int_0^{2\pi}M_\beta(w)\,|\psi(w)|\,d\theta.$$

Since the continuous linear forms δ_w^β defined by

$$\langle \psi, \delta_w^\beta \rangle = M_\beta(w)\psi(w)$$

be contained in the polar of the 0—neighborhood

$$V=\{\psi \in Z_\Omega\{M_A\};\ \|\psi\|_{\beta,H} \leqq 1\}.$$

We can define a positive Radon measure μ on V^0 by the following equality:

$$\int_{V^0} \Phi(a)d\mu = \frac{C_\alpha(\rho+\varepsilon)}{2\pi\varepsilon}\int_0^{2\pi} \Phi(\delta_w^\beta)d\theta \qquad \text{for } \Phi \in \mathcal{C}(V^0).$$

Therefore, for all $\psi \in Z_\Omega\{M_A\}$

$$\|\psi\|_{\alpha,K} \leqq \int_{V^0} |\langle \psi, a\rangle|\, d\mu.$$

But now for any compact subset K of Ω there exist finite many compact subsets K_i $(i=1, \cdots, n)$ which are considered as above and

$$K=\bigcup_{i=1}^n K_i.$$

By the similar definition as above for any $\alpha \in A$ there exist positive Radon measures μ_i and 0-neighborhoods V_i such that

$$\|\psi\|_{\alpha,K_i} \leqq \int_{V_i^0} |\langle \psi, a\rangle|\, d\mu_i \qquad \text{for all } \psi \in Z_\Omega\{M_A\}.$$

Here we put $V=\bigcap_{i=1}^n V_i$ then we can define the positive Radon measure μ on V^0 by the following:

$$\int_{V^0} \Phi(a)d\mu = \sum_{i=1}^n \int_{V_i^0} \Phi(a)d\mu_i \qquad \text{for } \Phi \in \mathcal{C}(V^0)$$

then for any $\psi \in Z_\Omega\{M_A\}$ we obtain

$$\|\psi\|_{\alpha,K} \leqq \sum_{i=1}^n \|\psi\|_{\alpha,K_i} \leqq \sum_{i=1}^n \int_{V_i^0} |\langle \psi, a\rangle|\, d\mu_i = \int_{V^0} |\langle \psi, a\rangle|\, d\mu.$$

Hence, by Theorem 1 in [3], $Z_\Omega\{M_A\}$ is a nuclear space.

Example. Let us denote by $\mathcal{A}(G)$ the linear space of all analytic functions in a bounded open subset G of C^n, then on it we can introduce a locally convex topology by semi-norm

$$\|\psi\|_K = \sup\{|\psi(z)| : z \in K\}$$

where K runs through all compact subset of G.

By setting $M_\alpha(z)=1$ for all $\alpha \in A$, $\mathcal{A}(G)$ becomes a $Z_G\{M_A\}$ space and it satisfies (N_1^0) and (N_2^0), therefore $\mathcal{A}(G)$ is a nuclear space.

References

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