27. Some Generalizations of QF-Rings

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1. Introduction. Throughout this paper all notations and all terminologies are the same as in T. Kato [5].

Recently there have been developed nice generalizations of QF-rings. B. L. Osofsky [6] has studied rings R for which R is an injective cogenerator in the category of right R-modules $\mathcal{M}_R$. Osofsky's theorem [6, Theorem 1] states that, if R is an injective cogenerator in $\mathcal{M}_R$, then R modulo its Jacobson radical J is Artinian. G. Azumaya [1] and Y. Utumi [8] have independently characterized rings R for which every faithful left R-module is a generator in ${}_R\mathcal{M}$. Such rings are called left PF. A theorem of Azumaya-Utumi states that a ring R is left PF if and only if R is left self-injective, R/J is Artinian, and every nonzero left ideal contains a simple one. T. Kato [4], [5] has studied rings R for which the injective hull $E(R_R)$ of R_R is torsionless and has proved the equivalence of the following statements:

(1) *R is right PF.*

(2) *R is an injective cogenerator in $\mathcal{M}_R$.*

(3) *$E(R_R)$ is torsionless and R is an S-ring.*

(4) *R is a cogenerator in $\mathcal{M}_R$ and is a right S-ring.*

In this paper we shall be concerned with the following condition:

(a) if U is a simple right (resp. left) ideal of a ring R, then there exists $a \in R$ such that $U \approx aR$, $E(aR) \subset R$ (resp. $U \approx Ra$, $E(Ra) \subset R$).

2. The condition (a). Proposition 1. *The following conditions on a ring R are equivalent:*

(1) *R satisfies* (a) *for simple right ideals.*

(2) *$E(U)$ is torsionless for each simple right ideal U.*

Proof. (1) implies (2) trivially.

(2) implies (1). Let U be a simple right ideal. Since $E(U)$ is torsionless by assumption, we have a map f: $E(U) \to R$ such that $U \to E(U) \to R$ is nonzero, or equivalently, a monomorphism by T. Kato [5, (1.1)]. f must be a monomorphism since $E(U)' \supset U$. From this our conclusion (1) follows immediately.

In my previous paper [5], we have discussed rings R for which $E(R_R)$ is torsionless. In the following we shall compare such rings with rings satisfying (a).

Proposition 2. *Let $E(R_R)$ be torsionless. Then R satisfies* (a) *for simple right ideals.*

Proof. Since $E(R_R)$ is torsionless, the injective hull of every torsionless right R-module is torsionless by T. Kato [5, Prop. 1]. Thus R satisfies (a) for simple right ideals by Proposition 1.

The following proposition is known, and we omit the proof.

Proposition 3. *The following conditions are equivalent for any ring R:*

(1) *R is a cogenerator in $\mathcal{M}_R$.*

(2) *R satisfies* (a) *for simple right ideals and is a left S-ring.*

The following lemma is useful in this paper (see K. Sugano [7, Lemma 3]).

Lemma 1. *If aR, $a \in R$, is a simple right ideal such that $E(aR) \subset R$, then Ra is a unique simple left ideal in $l(r(a))$.*

Proof. Let $0 \neq b \in l(r(a))$. Then $r(a)=r(b)$ by the maximality of r(a), and hence the mapping $br \rightarrow ar$, $r \in R$, gives a homomorphism of bR onto aR. Since $E(aR) \subset R$, this map is given by the left multiplication of an element of R. Thus $Ra \subset Rb$. This shows that Ra is a unique simple left ideal in $l(r(a))$.

Corollary. *Let R satisfy* (a) *for simple right ideals, and U a simple right ideal. Then U^* contains a unique simple submodule.*

Proof. Take $a \in R$ such that $U \approx aR$, $E(aR) \subset R$. Then $U^* \approx (aR)^* \approx (R/r(a))^* \approx l(r(a))$. Hence U^* contains a unique simple submodule by the above lemma.

We have seen in T. Kato [5, Lemma 2] the following lemma which is also useful.

Lemma 2. *The following conditions on a ring R are equivalent:*

(1) *The dual of any simple left R-module is zero or simple.*

(2) *The dual of any simple left ideal of R is simple.*

(3) *If Ra, $a \in R$, is simple then $r(l(a))=aR$.*

(4) $\mathrm{Ext}^1_R(R/U, R)=0$ *for each simple left ideal U.*

If R is a cogenerator in $\mathcal{M}_R$, then R satisfies (a) for simple right ideals by Proposition 3 and $\mathrm{Ext}^1_R(R/U, R)=0$ for each simple left ideal U by Lemma 2. This observation shows that the following theorem is applicable to right self-cogenerator rings.

Theorem 1. *Let R satisfy* (a) *for simple right ideals, and let* $\mathrm{Ext}^1_R(R/U, R)=0$ *for each simple left ideal U. Then*

(1) *The mapping*

$$Ra \rightarrow aR, \quad a \in R$$

gives a one-to-one, onto, correspondence between isomorphism classes of simple left ideals and isomorphism classes of simple right ideals.

(2) *Each simple left ideal is of the form Re/Je, $e=e^2 \in R$.*

Proof. (1) We first show that our correspondence is well defined. In fact, let $Ra \approx Rb$ be simple left ideals. Then

$$aR = r(l(a)) \approx (R/l(a))^* \approx (Ra)^* \approx (Rb)^* \approx r(1(b)) = bR$$

is simple by Lemma 2.

[onto] Let U be any simple right ideal. By virtue of (a), $U \approx aR$, $E(aR) \subset R$, for some $a \in R$. Then Ra is simple by Lemma 1, and $Ra \rightarrow aR \approx U$.

[one-to-one] Let Ra, Rb, be simple such that $aR \approx bR$. Then

$$l(r(a)) \approx (aR)^* \approx (bR)^* \approx l(r(b)),$$

and Ra, Rb, are simple submodules of $l(r(a)), l(r(b))$, respectively. Therefore $Ra \approx Rb$ by Corollary to Lemma 1.

(2) Let U be a simple left ideal. Then U^* is simple by Lemma 2. By the condition (a), $U^* \approx aR$, $eR = E(aR)$, for some $a, e = e^2 \in R$. We show that $aR = er(J)$. In fact, Ra is simple by Lemma 1 and hence $Ja = 0$, or equivalently, $a \in r(J)$. Thus $aR \subset er(J)$ since $aR \subset E(aR) = eR$. Next, Re/Je is simple since $eR = E(aR)$ is indecomposable injective (see B. L. Osofsky [6, Lemma 3]). Then $er(J) \approx (Re/Je)^*$ is simple by Lemma 2. Thus we have $aR = er(J)$. Now, $U, Re/Je$, are the unique simple submodules of U^{**}, $(aR)^* \approx (er(J))^* \approx (Re/Je)^{**}$, respectively by Corollary to Lemma 1 and by the fact that both U and Re/Je are torsionless. Therefore $U \approx Re/Je$ since $U^{**} \approx (aR)^*$.

The statement (2) in the preceding theorem is meaningful by virtue of the following lemma which will be of interest by itself.

Lemma 3. *The following conditions on a ring R are equivalent:*

(1) *R is semi-simple.*

(2) *R is a right S-ring with zero Jacobson radical.*

(3) *Each simple left R-module is projective.*

Proof. (1)$\Rightarrow$(2) is evident.

(2) implies (3). Let U be any simple left R-module. We may assume, without loss of generality, that U is a simple left ideal of R since R is a right S-ring. But, since rad $R = 0$, U is generated by an idempotent (see N. Jacobson [3, p. 57]) and hence U is projective.

(3) implies (1). It suffices to show that R equals its left socle, say, S. Assume $R \neq S$. Then $S \subset L$ for some maximal left ideal L. Since R/L is projective by assumption, $R = L \oplus L'$ for some left ideal L'. Now $L' \approx R/L$ is simple, and hence $L' \subset S \subset L$. But this contradicts the fact that $L \cap L' = 0$.

We are now ready for one of our main results.

Theorem 2. *The following conditions on a ring R are equivalent:*

(1) *R is an injective cogenerator in $\mathcal{M}_R$.*

(2) *R satisfies* (a) *for simple right ideals,* $\mathrm{Ext}^1_R(R/U, R) = 0$

for each simple left ideal U, and R is a right S-ring.

Proof. (1) implies (2). In view of Proposition 3 and Lemma 2, it is enough to show that, if R is an injective cogenerator in $\mathcal{M}_R$, then R is a right S-ring. Let R be an injective cogenerator in $\mathcal{M}_R$. Then R/J is Artinian by B. L. Osofsky [6, Theorem 1]. Hence, by virtue of Theorem 1 (1) together with the fact that R is a left S-ring, we conclude that R is a right S-ring (see [4, Theorem 6]).

(2) implies (1). Assume (2). Since R is a right S-ring, each simple left R-module is isomorphic to a simple left ideal. Hence each simple left R-module is of the form Re/Je, $e=e^2\in R$, by Theorem 1 (2). Thus each simple left R-module is R/J-projective and hence R/J is Artinian by Lemma 3. Since R/J is Artinian and R is a right S-ring, R is a left S-ring by Theorem 1 (1). Consequently R is a cogenerator in $\mathcal{M}_R$ by Proposition 3. Now the right self-injectivity of R follows from T. Kato [5, Theorem 1].

Let R satisfy (a) for simple left ideals, and U a simple left ideal. Then by (the left-right analogy of) Corollary to Lemma 1, U^* contains a unique simple submodule, and this submodule is regarded as a simple right ideal. We shall use this fact to show the following theorem which is analogous to Theorem 1.

Theorem 3. *Let R satisfy* (a) *for each simple one-sided ideal.*

(1) *The mapping*

$$U \to \text{the unique simple submodule of } U^*$$

gives a one-to-one, onto, correspondence between isomorphism classes of simple left ideals and isomorphism classes of simple right ideals.

(2) *Each simple left ideal is of the form Re/Je, $e=e^2\in R$.*

Proof. (1) Let U be a simple left ideal. By virtue of (a), $U \approx Ra$, $E(Ra)\subset R$, for some $a\in R$. Then our correspondence is just

$$U\approx Ra\to aR,$$

since aR is the unique simple submodule of $[r(l(a))\approx(Ra)^*\approx U^*]$ by (the left-right analogy of) Lemma 1.

[one-to-one] Let Ra, Rb, be simple left ideals such that $E(Ra)$, $E(Rb)\subset R$. Assume $aR\approx bR$. Then $l(r(a))\approx(aR)^*\approx(bR)^*\approx l(r(b))$. But, Ra, Rb, are simple submodules of $l(r(a))$, $l(r(b))$, respectively. Hence $Ra\approx Rb$ by Corollary to Lemma 1.

[onto] Let V be any simple right ideal. Take $a\in R$ such that $V\approx aR$, $E(aR)\subset R$, making use of (a). Then Ra is simple by Lemma 1 and

$$Ra\to \text{the unique simple submodule of } [(Ra)^*\approx r(l(a))]\approx aR\approx V.$$

(2) Let U be a simple left ideal. By virtue of (a), $U\approx Ra$, $E(Ra)\subset R$, for some $a\in R$. Then aR is simple. Choose $b\in R$ such that $aR\approx bR$, $eR=E(bR)$, $e=e^2\in R$, making use of (a). Since eR is

injective indecomposable, Re/Je is simple. Now, Re/Je is isomorphic to a simple left ideal U', say, since $0\neq(Re/Je)^*\approx er(J)\supset bR$. Since

$$Re/Je\approx U'\rightarrow \text{the unique simple submodule of } [U'^*\approx(Re/Je)^* \approx er(J)]\approx bR,$$
$$U\approx Ra\rightarrow aR\approx bR,$$

we have $U\approx U'\approx Re/Je$ by our one-to-one correspondence.

Making use of Theorem 3, we can now establish the following refinement of a portion of T. Kato [5, Cor. to Theorem 1].

Theorem 4. *The following conditions on a ring R are equivalent:*

(1) *R is an injective cogenerator both in ${}_R\mathcal{M}$ and in $\mathcal{M}_R$.*

(2) *$E({}_RR)$ and $E(R_R)$ are torsionless and R is a right S-ring.*

(3) *$E(U)$ and $E(V)$ are torsionless for any simple left R-module U and any simple right ideal V.*

Proof. (1) trivially implies (2).

(2) implies (3). Let U, V, be a simple left R-module and a simple right ideal respectively. Since R is a right S-ring, U is isomorphic to a simple left ideal. Thus $E(U)\subset E({}_RR)$ and $E(V)\subset E(R_R)$ are torsionless.

(3) implies (1). Since $E(U)$ is torsionless for any simple left R-module U, R is a cogenerator in ${}_R\mathcal{M}$ by [5, Prop. 3], and hence R is a right S-ring. Furthermore R satisfies (a) for each simple one-sided ideal by Proposition 1. Now apply Theorem 3 and we conclude that R/J is Artinian and that R is a left S-ring along the same lines as in the proof of Theorem 2. Thus R is an injective cogenerator both in ${}_R\mathcal{M}$ and in $\mathcal{M}_R$.

3. *QF*-rings. A ring R is called QF if R is both right and left self-injective and R is both right and left Artinian. In the following we give a short proof of a result due to S. Eilenberg and T. Nakayama [2, Theorem 18].

Theorem 5. *The following conditions on a ring R are equivalent:*

(1) *R is QF.*

(2) *R is right self-injective and right Artinian.*

(3) *R is right self-injective and left Artinian.*

(4) $\mathrm{Ext}^1_R(R/U, R)=0$ *for each simple one-sided ideal U, and R is right* (*or left*) *Artinian.*

Proof. (1)⇒(2), (3) is trivial.

(2) implies (4). The first part of the condition (4) follows at once from the fact that R is an injective cogenerator in $\mathcal{M}_R$.

(3) implies (4). By assumption, R is right self-injective, R/J is Artinian, and every right ideal $\neq 0$ contains a simple one. Hence R is an injective cogenerator in $\mathcal{M}_R$.

(4) implies (1). The first part of the condition (4) implies that the dual of each simple one-sided ideal is simple. Therefore R is QF by the same argument as in the proof of [5, Proposition 4].

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