76. *On the Compatibility of the AP- and the D-integrals*

By Yôto KUBOTA
Department of Mathematics, Ibaraki University

(Comm. by Kinjirô KUNUGI, M. J. A., May 13, 1968)

1. Introduction. We call two definitions of integration *compatible* if every function which is integrable in both senses is integrable to the same value in both senses. H. W. Ellis [3] has shown that the AP-integral and the CP-integral [2] are not compatible. Recently V. A. Skvorcov [5] established that, if $f(x)$ is CP-integrable with the CP-integral $F(x)$ as well as D-integrable with the D-integral $F_1(x)$ over $[a, b]$ then $F_1(x)=F(x)+C$ on $[a, b]$ where C is a constant. This assertion shows that the CP-integral do not contradict the general Denjoy integral.

The aim of this paper is to show *directly* that the D-integral and the AP-integral are compatible.

2. The AP-integral. A real valued function $f(x)$ is said to be $\underline{AC}$ on a linear set E if, to each positive number ε, there exists a number $\delta>0$ such that

$$\sum\{f(b_k)-f(a_k)\}>-\varepsilon$$

for all finite non-overlapping sequences of intervals $\{(a_k, b_k)\}$ with end points on E and such that

$$\sum(b_k-a_k)<\delta.$$

There is a corresponding definition $\overline{AC}$ on E. A function which is both $\underline{AC}$ and $\overline{AC}$ on E is termed AC on E. If the set E is the sum of a countable number of sets E_k on each of which $f(x)$ is $\underline{AC}$ then $f(x)$ is said to be $\underline{ACG}$ on E. If the sets E_k are assumed to be closed, then $f(x)$ is said to be $(\underline{ACG})$ on E. Similarly we can define $\overline{ACG}$ and $(\overline{ACG})$ on E. A function is said to be (ACG) on E if it is both $(\underline{ACG})$ and $(\overline{ACG})$ on E. A continuous function which is both $\underline{ACG}$ and $\overline{ACG}$ on E is termed ACG on E.

The function $M(x)$ is called an *AP-major* function of $f(x)$ in [a, b] if

(i) $M(a)=0$;

(ii) $M(x)$ is approximately continuous for all $x\in[a, b]$;

(iii) $\underline{AD}\ M(x)\geqq f(x)$ everywhere on $[a, b]$;

(iv) $\underline{AD}\ M(x)>-\infty$ everywhere on $[a, b]$.

The *AP-minor* function $m(x)$ is defined analogously.

A function $f(x)$ is called AP-integrable over $[a, b]$ if for its *AP-major* function and its *AP-minor* function the following equality is satisfied: $\inf_M M(b)=\sup_m m(b)$. This common value is taken as the value of the definite AP-integral $F(b)$ of $f(x)$ over $[a, b]$. Moreover, the indefinite integral $F(x)$ is defined, as $\inf M(x)=\sup m(x)$. This integral exists, since for any pair $M(x)$ and $m(x)$ the difference $M(x)-m(x)$ is non-decreasing ([1]). It is well known that the difference $M(x)-F(x)$ $[F(x)-m(x)]$ is a non-decreasing function.

The following lemmas will be needed.

Lemma 1. *If $f(x)$ is approximately continuous and $\underline{AD}\ f(x) > -\infty$ $[\overline{AD}\ f(x) < +\infty]$ everywhere on $[a, b]$ then $f(x)$ is $\underline{(ACG)}$ $[\overline{(ACG)}]$ on $[a, b]$.*

Proof. We need only consider the first case. Since this lemma has been essentially established by J. Ridder in the proof of Theorem 9 in [7], we sketch the proof.

Since $\underline{AD}\ f(x) > -\infty$ at each point x, there exists n for each x such that the set

$$E_x = \left\{t : \frac{f(t)-f(x)}{t-x} \geqq -n\right\}$$

has the point x as a point of density. Therefore, denoting by E_n the set of the points x such that the inequality $0 \leqq h \leqq 1/n$ implies

$$m(E_x[x-h, x+h]) > 3/2 \cdot h,$$

we have $[a, b] = \bigcup_{n=1}^{\infty} E_n$. Moreover let

$$E_{ni} = [i/n, (i+1)/n] \cap E_n$$

for every integer i. We first show that $f(x)$ is $\underline{AC}$ on each E_{ni}. Next we must show that $f(x)$ is also $\underline{AC}$ on the closure $\overline{E}_{ni}$. For this purpose we put $g_n(x) = f(x) + nx$ and prove by using the approximate continuity of $g_n(x)$ that

$$\lim g_n(x) = g_n(\xi) \quad (x \to \xi,\ x \in E_{ni})$$

for any limiting point ξ of $\overline{E}_{ni}$. This, together with the fact that $f(x)$ is $\underline{AC}$ on E_{ni}, implies the lower absolute continuity $\underline{AC}$ of $f(x)$ on $\overline{E}_{ni}$. Since $[a, b] = \bigcup_n \bigcup_i \overline{E}_{ni}$, the lemma is proved.

Lemma 2. *If $f(x)$ is AP-integrable on $[a, b]$ then its indefinite AP-integral $F(x)$ is (ACG) on $[a, b]$.*

Proof. Since $f(x)$ is AP-integrable on $[a, b]$ there exists a sequence of major functions $\{M_k(x)\}$ and a sequence of minor functions $\{m_k(x)\}$ such that

$$(1) \qquad \lim M_k(b) = F(b) = \lim m_k(b).$$

Since $M_k(x)-F(x)$ and $F(x)-m_k(x)$ are non-decreasing, it holds that

$$(2) \qquad \lim M_k(x) = F(x) = \lim m_k(x) \quad \text{for } a \leqq x \leqq b.$$

By Lemma 1, any $M_k(x)[m_k(x)]$ is $(\underline{ACG})[(\overline{ACG})]$ on $[a, b]$, so that the interval $[a, b]$ is expressible as the sum of a countable number of closed sets E_k such that any M_k is $\underline{AC}$ on any E_k and at the same time any m_k is $\overline{AC}$ on any E_k. It is sufficient to prove that $F(x)$ is AC on any E_k. For this purpose we shall show that $F(x)$ is both $\underline{AC}$ and $\overline{AC}$ on E_k.

Suppose that $F(x)$ is not $\underline{AC}$ on E_k. Then there exists an $\varepsilon > 0$ and a finite sequence of non-overlapping intervals $\{(a_\nu, b_\nu)\}$ with end points on E_k such that

$$\sum(b_\nu - a_\nu) < \delta$$

but

$$\sum\{F(b_\nu) - F(a_\nu)\} \leqq -\varepsilon. \tag{3}$$

Since we can find a natural number p such that

$$M_p(b) - F(b) \leqq 1/2 \cdot \varepsilon$$

and since $M_p(x) - F(x)$ is non-decreasing on $[a, b]$, we have

$$\begin{aligned} &\sum\{M_p(b_\nu) - M_p(a_\nu)\} - \sum\{F(b_\nu) - F(a_\nu)\} \\ &= \sum[\{M_p(b_\nu) - F(a_\nu)\} - \{M_p(a_\nu) - F(a_\nu)\}] \\ &\leqq M_p(b) - F(b) \leqq 1/2 \cdot \varepsilon. \end{aligned} \tag{4}$$

It follows from (3) and (4) that

$$\begin{aligned} \sum\{M_p(b_\nu) - M(a_\nu)\} &\leqq \sum\{F(b_\nu) - F(a_\nu)\} + 1/2 \cdot \varepsilon \\ &\leqq -1/2 \cdot \varepsilon. \end{aligned}$$

This contradicts the fact that $M_p(x)$ is $\underline{AC}$ on E_k. Hence $F(x)$ is $\underline{AC}$ on E_k.

Similarly we can prove that $F(x)$ is $\overline{AC}$ on E_k. Thus $F(x)$ is (ACG) on $[a, b]$.

3. Compatibility of the AP- and the D-integrals. The author [6] has shown the following lemma in defining the AD-integral, which will play an essential role to our problem.

Lemma 3. *If $f(x)$ is approximately continuous, (ACG) on $[a, b]$ and if $AD\ f(x) = 0$ a.e. then $f(x)$ is constant on $[a, b]$.*

Theorem. *The AP-integral and the D-integral are compatible.*

Proof. Let $f(x)$ be AP-integrable with the AP-integral $F(x)$ as well as D-integrable with the D-integral $F_1(x)$ over $[a, b]$. We consider the difference $F_2(x) = F(x) - F_1(x)$ and show that $F_2(x) =$ constant on $[a, b]$ which implies the compatibility of the AP- and the D-integrals.

The function $F(x)$ is approximately continuous on $[a, b]$ and $AD\ F(x) = f(x)$ a.e. ([2], p. 276). It is also (ACG) by Lemma 2. By the descriptive difinition of the D-integral, $F_1(x)$ is ACG on $[a, b]$ and $AD\ F_1(x) = f(x)$ a.e. It is well known ([4], p. 224) that, if a function is AC on E and is continuous on its closure $\bar{E}$ then it is also AC on $\bar{E}$.

Hence $F_1(x)$ is also (ACG) on $[a, b]$. It follows that $F_2(x)$ is approximately continuous, (ACG) on $[a, b]$ and $AD\ F_2(x)=0$ a.e. By Lemma 3, we have $F_2(x)=$constant, which proves the theorem.

References

[1] J. C. Burkill: The approximately continuous Perron integral. Math. Zeit., **34**, 270–278 (1931).

[2] ——: The Cesàro-Perron integral. Proc. London Math. Soc., **34**, 314–322 (1932).

[3] H. W. Ellis: On the compatibility of the approximate Perron and the Cesàro Perron integrals. Proc. Amer. Math. Soc., **2**, 396–397 (1951).

[4] S. Saks: Theory of the Integral. Warsaw (1937).

[5] V. A. Skvorcov: Some properties of the *CP*-integral. Mat. Sb., **60**, 304–324 (1963).

[6] Y. Kubota: An integral of the Denjoy type. III. Proc. Japan Acad., **43**, 441–444 (1967).

[7] J. Ridder: Ueber die gegenseitigen Beziehungen verschieder approximativ stetiger Denjoy-Perron integrale. Fund. Math., **22**, 136–162 (1934).