68. *A Minimal Property for an Operator of Hilbert-Schmidt Class*

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1. If T is a completely continuous operator defined on a Hilbert space H, then T can be expressed in the *Schatten formula*:

$$T=\sum_{i=1}^{\infty}\lambda_i\varphi_i\otimes\psi_i, \tag{1}$$

where (i) $\{\lambda_i\}$ is a decreasing sequence of positive numbers which are proper values of

$$|T|=(T^*T)^{\frac{1}{2}}, \tag{2}$$

(ii) $\{\varphi_i\}$ and $\{\psi_i\}$ are orthonormal sets in H, and (iii) a *dyad* $f\otimes g$ is defined by

$$(f\otimes g)h=(h\,|\,g)f, \tag{3}$$

for every $h\in H$, cf. [2]. Since the proper values of a completely continuous operator $|T|$ converge to zero, the series of (1) converges uniformly.

An operator T acting on H is of *Hilbert-Schmidt class* if

$$||T||_2^2=\sum_{i=1}^{\infty}||T\phi_i||^2 \tag{4}$$

is finite whenever $\{\phi_i\}$ is a orthonormal base of H. An operator T of Hilbert-Schmidt class is completely continuous and

$$||T||_2^2=\sum_{i=1}^{\infty}\lambda_i^2, \tag{5}$$

where $\{\lambda_i\}$ is the coefficients of the Schatten formula (1).

The purpose of the present note is to show the following minimal property of the Schatten formula:

Theorem 1. *If T is of Hilbert-Schmidt class and expressed in (1), then*

$$||T-\lambda_1\varphi_1\otimes\psi_1||_2$$

attains its minimum among all approximation by dyads: that is,

$$||T-\lambda_1\varphi_1\otimes\psi_1||_2\leqq||T-f\otimes g||_2, \tag{6}$$

for every dyad $f\otimes g$.

2. Let $H=L^2[0,1]$. If $u(x,y)$ is a measurable function defined on $[0,1]\times[0,1]$ with

$$||u||^2=\int_0^1\int_0^1|u(x,y)|^2dx\,dy<+\infty,$$

then, for every $f\in H$,

(7) $$Tf(y)=\int_0^1 u(x, y)f(x)dx$$

defines and operator T on H which is of Hilbert-Schmidt class with $\|T\|_2=\|u\|$. If g and h are functions in H and

(8) $$u(x, y)=g(x)^*h(y),$$

then

$$Tf(y)=\int_0^1 h(y)g(x)^*f(x)dx=(f\mid g)h(y)$$

implies that u of (8) defines a dyad on H.

S. Hitotumatu pointed out, in his recent study [1] on the numerical approximation of a function of two variables by the product of functions, the following

Theorem 2. (Hitotumatu). *If $u(x, y)$ is square-integrable, then there are two square-integrable functions $g(x)$ and $h(y)$ such that*

$$\|u-g^*h\|^2=\int_0^1\int_0^1 |u(x, y)-g(x)^*h(y)|^2dx\, dy$$

attains its minimum.

By the equality of the norms, it is obvious that Theorem 1 implies Hitotumatu's theorem. Hitotumatu's proof of Theorem 2 is based on the weak compactness of the unit ball and the semi-continuity of the norm with respect to the weak topology, whereas our proof utilizes only the Schatten formula and the definition (4) of the Schmidt norm.

3. Suppose that T is a Hilbert-Schmidt operator and expressed in (1). Let α be a number and f, g elements of H with $\|f\|=\|g\|=1$. If $\{\phi_i\}$ is a base of H with $\phi_1=g$, then (4) implies

$$\begin{aligned}\|T-\alpha f\otimes g\|_2^2&=\sum_{i=1}^{\infty}\|(T-\alpha f\otimes g)\phi_i\|^2\\&=\sum_{i=1}^{\infty}\|T\phi_i-\alpha(\phi_i\mid g)f\|^2\\&=\sum_{i=2}^{\infty}\|T\phi_i\|^2+\|T\phi_1-\alpha f\|^2\\&=\|T\|_2^2-\|T\phi_1\|^2+\|T\phi_1-\alpha f\|^2.\end{aligned}$$

Hence, we have

(9) $$\|T-\alpha f\otimes g\|_2^2=\|T\|_2^2-\|Tg\|^2+\|Tg-\alpha f\|^2.$$

To minimize the right hand side of (9), we need to maximize $\|Tg\|$ under $\|g\|=1$ and to minimize $\|Tg-\alpha f\|$ under $\|f\|=\|g\|=1$. The first is obviously solved by putting $g=\psi_1$, and the second is solved if $\alpha=\lambda_1$ and $f=\varphi_1$ since

$$T\psi_1=\sum_{i=1}^{\infty}\lambda_i(\psi_1\mid\psi_i)\varphi_i=\lambda_1\varphi_1.$$

Therefore, (6) is satisfied.

4. The residue of the approximation by a dyad is now easily computed:

$$||T-\lambda_1\varphi_1\otimes\psi_1||_2^2=\sum_{i=2}^{\infty}\lambda_i^2.$$

Hence, if $|T|$ has only one non-zero proper value of multiplicity one, then the approximation is exact, that is, T is a dyad.

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References

[1] S. Hitotumatu: On the approximation of a function of two variables by the product (in Japanese). Zyoho Syori (Data Processing), **9**, 14–17 (1968).

[2] R. Schatten: Norm Ideals of Completely Continuous Operators. Ergeb. Math. Grenzgeb., N. F., Hf., **27**, Springer, Berlin-Göttingen-Heidelberg (1960).