100. On a Closed Graph Theorem for Topological Groups

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Let $\mathcal{C}$ be a class of Hausdorff topological groups satisfying the following: If for any Hausdorff topological group H there is a continuous almost open homomorphism of some member $G \in \mathcal{C}$ onto H then $H \in \mathcal{C}$. Let $E \in \mathcal{C}$ and let F be a $B_r(\mathcal{C})$-group. Then each almost open almost continuous homomorphism of E onto F, the graph of which is closed in $E \times F$, is continuous.

This theorem is stated for "into" homomorphisms in [2] p. 94, Theorem 5. The proof given there requires ontoness, however. The purpose of this note is to extend this theorem for "into" homomorphisms. But then one requires, in addition, that the range group F is abelian. When both E and F are abelian, a special case of the above theorem can be proved very easily as has been done by Baker [1]. Baker's method of proof requires that E and F both are abelian. In this note, we weaken Baker's assumption by assuming that only F is abelian. The proof of this extension is a slight modification of the proof given in my book ([2], p. 94, Theorem 5).

We shall follow the notations and terminology of [2] without any specific mention.

We prove the following

Theorem. *Let $\mathcal{C}$ be a class of Hausdorff topological groups satisfying the following: For any arbitrary Hausdorff topological group H if there exists a continuous almost open homomorphism of some member in $\mathcal{C}$ into H, then H is also in $\mathcal{C}$. Let $E \in \mathcal{C}$ and F an abelian $B_r(\mathcal{C})$-group. Let f be an almost continuous almost open homomorphism of E into F, the graph of which is closed in $E \times F$. Then f is continuous.*

Proof. Let u denote the initial topology on F. Let v denote the topology defined by the sets

$$U^* = \overline{f(\overline{f^{-1}(U)})},$$

where U runs over a fundamental system of neighborhoods of the identity of F, as in Theorem 5 ([2], p. 94). It is shown there that v is a Hausdorff topology such that F_v is a topological group. (Observe that "ontoness" of f is needed to show that $U^* \supset aU_1^* a^{-1}$. However, if F is abelian then this condition is automatic and therefore "ontoness" of f is unnecessary.)

Since f is not assumed to be onto, it is not possible to say that $U^* \supset U$ and therefore it may not be possible to conclude that u is finer than v. Now to get a topology which is coarser than u, we define another topology w defined by the sets $\{UU^*\}$, where U, as before, runs over a fundamental system of neighborhoods of the identity of F. Now remembering that F is abelian, it is easy to check that Conditions (a) and (b) of Theorem 3, p. 46 [2] are satisfied. Hence F_w is a topological group. To show that w is a Hausdorff topology, we use closed graph, as for v.

For each U in $\{U\}$, a base of symmetric closed u-neighborhood of the identity of F, there exists $U_1 \in \{U\}$ such that $U_1^3 \subset U$. We wish to show that $\bigcap UU^* = \{e'\}$, e'=identity of F. Let $y \in UU^*$ for each $U \in \{U\}$, then $y \in U_1U_1^* = U_1\overline{f(\overline{f^{-1}(U_1)})}$, and hence there exist $x_1 \in \overline{f^{-1}(U_1)}$, $u_1 \in U_1$ such that $u_1 f(x_1) \in yU_1$. But $\overline{f^{-1}(U_1)} \subset Vf^{-1}(U_1)$, where V is an arbitrary neighborhood of the identity e of E. Hence $x_1 \in Vf^{-1}(U_1)$. That means that there exists $x_2 \in V$ such that $x_2^{-1}x_1 \in f^{-1}(U_1)$. Since f is a homomorphism, it follows that $f(x_2) \in f(x_1)U_1$. Thus

$$f(x_2) \in u_1^{-1}yU_1U_1 \subset yU_1U_1^2 \subset yU.$$

Since $(x_2, f(x_2))$ is in the graph of f which is closed, and also $(x_2, f(x_2)) \in (V, yU)$, where V and U are arbitrary neighborhoods of the identities of E and F respectively, it is clear that $f(e)=e'=y$. Thus w is a Hausdorff topology.

Since $UU^* \supset U$, it is clear that u is finer than w. Therefore $u \supset w(u) \supset w$, (see the proof of Theorem 5, and the notations in Proposition 8, §31, of [2].) We show that $w(u)=w$. For this it is sufficient to show that $w(u) \subset w$.

Let U_1, U be in $\{U\}$ such that $U_1^2 \subset U$. We show that $U_1U_1^* \subset Cl_w U$.

Let $y \in U_1U_1^*$ and let W be an arbitrary member of $\{U\}$. Since $y \in U_1U_1^* = U_1\overline{f(\overline{f^{-1}(U_1)})}$, there exist $x_1 \in \overline{f^{-1}(U_1)}$ and $u_1 \in U_1$ such that $u_1 f(x_1) \in yW$. Since f is almost continuous, $\overline{f^{-1}(W)}$ is a neighborhood of the identity in E. Hence

$$x_1 \in \overline{f^{-1}(U_1)} \subset f^{-1}(U_1)\overline{f^{-1}(W)}.$$

That means there is $x_2 \in f^{-1}(U_1)$ such that

$$x_2^{-1}x_1 \in \overline{f^{-1}(W)}.$$

Since f is a homomorphism and since W^* is symmetric,

$$f(x_2) \in f(x_1)\overline{f(\overline{f^{-1}(W)})} \subset f(x_1)W^*.$$

But $u_1 f(x_1) \in yW$ and so we have

$$u_1 f(x_2) \in yWW^* \cap U,$$

because $U_1^2 \subset U$. Since y is arbitrary in $U_1U_1^*$, we have shown that

$U_1U_1^* \subset Cl_w U$. This proves that $w(u)=w$.

Now using the same arguments as in Theorem 5, p. 95 [2], we finish the proof.

Theorem 2. *Let E be any Hausdorff topological group and F an abelian $B_r(\mathcal{A})$-group. Let f be an almost continuous homomorphism of E into F, the graph of which is closed in $E \times F$. Then f is continuous.*

Proof. The proof of this Theorem is the same as that of Theorem 6, p. 96 [2], with the appropriate changes in topology, i.e., one considers w as defined above instead of v as defined in Theorem 5 on p. 95, [2].

This theorem extends the result of Baker [1] where he proves this under the conditions that both E and F are abelian.

References

[1] W. J. Baker: Topological groups and closed graph theorem. J. London Math. Soc., **42**, 217–225 (1967).
[2] T. Husain: Introduction to Topological Groups. W. B. Saunders Co., Philadelphia (1966).