94. On a Hardy's Theorem

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(Comm. by Zyoiti SUETUNA, M. J. A., June 12, 1968)

1. Introduction and theorems. 1.1. Let f be an even and integrable function with period 2π and with mean value zero and let its Fourier series be

$$f(x) \sim \sum_{n=1}^{\infty} a_n \cos nx. \tag{1}$$

We suppose always $1<p<\infty$. By L^p we denote the space of such functions whose p-th powers are integrable. We put

$$A_n = \frac{1}{n} \sum_{k=1}^{n} a_k \qquad (n=1, 2, \cdots), \tag{2}$$

then Hardy [1] proved that there is an integrable function F such that

$$F(x) \sim \sum_{n=1}^{\infty} A_n \cos nx. \tag{3}$$

Further he [1] proved the following

Theorem I. $f \in L^p \Rightarrow F \in L^p$.

Petersen [2] has proved that the space L^p in Theorem I can be replaced by the Lorentz space Λ^p [3] which consists of even and integrable functions f with mean value zero such that

$$\int_0^{\pi} f^*(t) t^{-1/q} dt < \infty \qquad (1/p + 1/q = 1),$$

where f^* is the monotone decreasing rearrangement of $|f(t)|$. It is known that $\Lambda^p \subset L^p$ ([3], p. 39). Petersen's theorem[1] is as follows:

Theorem II. $f \in \Lambda^p \Rightarrow F \in \Lambda^p$.

1.2. *Let L_0^p be the space of even and integrable functions f with mean value zero and with neighbourhood of the origin where the p-th power of $|f|$ is integrable.* Then Theorem I is generalized as follows:

Theorem I'. $f \in L_0^p \Rightarrow F \in L^p$.

We introduce another space M^p which consists of even and integrable functions f with mean value zero, satisfying the condition

$$\int_0^{\pi} |f(t)| t^{-1/q} dt < \infty \qquad (1/p + 1/q = 1),$$

(cf. [4]). Evidently $M^p \supset M^{p'}$ for $1<p<p'$. By Hölder's inequality we get

1) Petersen has proved similar theorems for the other Lorentz spaces.

(4) $$L^{p'}\subset M^p \quad \text{for all} \quad p'>p>1.$$

Proposition 1. $L_0^p \not\supset M^p \not\supset L^p$.

Concerning M^p, we have the following analogue of Theorem I.

Theorem 1. *$f\in M^p \Rightarrow F\in M^p\cap L^p$, where F is defined by* (2), (3).

As consequence of Theorem 1 and Proposition 1, we get:

Proposition 2. Converse of Theorem I does not hold, that is, there is a function f such that $F\in L^p$, but f does not (cf. [5]).

Proposition 3. Hardy's theorem cannot be strengthened as follows:

$$f\in L^p \Rightarrow F\in L^p\cap M^p.$$

Proposition 4. Converse of Theorem 1 does not hold, that is, there is a function f such that $F\in M^p\cap L^p$, but f does not belong to M^p.

If we denote by O^p the space of even and integrable functions f with mean value zero and satisfying the condition

$$f(t)=O(t^{-1/p}) \quad \text{as} \quad t\to 0,$$

then we get

Theorem 2. *$f\in L^p \Rightarrow F\in L^p\cap O^p$.*

As a corollary of Theorems 1 and 2, we get

$$f\in L^p\cap M^p \Rightarrow F\in L^p\cap M^p\cap O^p.$$

Proposition 5. Converse of Theorem 2 does not hold.

1.3. We shall now introduce another space N^p which consists of functions f, even, integrable, with mean value zero and satisfying the condition that the Cauchy integral

$$\int_{+0}^{\pi} f(t)t^{-1/q}dt=\lim_{\varepsilon\downarrow 0}\int_{\varepsilon}^{\pi} f(t)t^{-1/q}dt \qquad (1/p+1/q=1)$$

exists finitely. Evidently $M^p\subset N^p$. We have

Theorem 3. *$f\in N^p \Rightarrow F\in N^p$.*

Proposition 6. Theorem 3 cannot be strengthened as follows:

$$f\in N^p \Rightarrow F\in N^p\cap L^p.$$

Proposition 7. Converse of Theorem 3 does not hold, that is, there is a function f such that $F\in N^p$, but f does not.

1.4. Bellmann [6] has proved the following dual of Theorem I:

Theorem III. *$f\in L^p \Rightarrow G\in L^p$, where*

(5) $$G(x)\sim \sum_{n=1}^{\infty} B_n\cos nx, \qquad B_n=\sum_{k=n}^{\infty}(a_k/k).$$

We can prove the dual of Theorems 1 and 2, that is:

Theorem 4. *$f\in M^p \Rightarrow G\in L^p\cap M^p$,*

Theorem 5. *$f\in L^p \Rightarrow G\in L^p\cap O^p$.*

Combining Theorem 4 and Proposition 1, we get:

Proposition 8. Converse of Theorem III does not hold.

Proposition 9. Converses of Theorems 4 and 5 do not hold.

Proposition 10. Dual of Theorem 3 does not hold, that is, there is a function f such that $f \in N^p$, but G does not.

1.5. We put $F = Tf$, then we get, collecting above results,

$$L_0^p \xrightarrow{T} L^p \xrightarrow{T} L^p \cap O^p, \quad M^p \xrightarrow{T} L^p \cap M^p \xrightarrow{T} L^p \cap M^p \cap O^p,$$

where $A \xrightarrow{T} B$ means that T maps the set A into a proper subset of B. Further, by T^n we denote the n-th iteration of T, then we get

Theorem 6. *For* $f \in L_0^p \cup M^p$, $\lim_{n\to\infty} \|T^n f\|_{L^p} = 0$ or ∞ *according as* $a_1(f) = 0$ *or* $a_1(f) \neq 0$. Therefore, if we put $S_1 = (f\,;\, f \in L_0^p \cup M^p, a_1(f) = 0)$ and $S_2 = (L_0^p \cup M^p) - S_1$, then $\lim_{n\to\infty} T^n S_1 = (0)$, where (0) denotes the set of almost everywhere vanishing functions and $\lim_{n\to\infty} \|T^n f\|_{L^p} = \infty$ for every $f \in S_2$.

2. Proof of Proposition 1. Let $f_1(t)$ be the even and periodic function such that

$$f_1(t) = t^{-1/p}\left(\log \frac{2\pi}{t}\right)^{-1} - A_1 \quad \text{on} \quad (0, \pi) \tag{6}$$

where the constant A_1 is taken as the mean value of f_1 is zero. Then $f_1 \in L^p$, but not in M^p. On the other hand, we take the even and periodic function f_2 defined on $(0, \pi)$ as follows:

$$\begin{aligned} f_2(t) &= 2^k k^{-2} - A_2 \quad \text{on} \quad (k^{-1}, k^{-1} + 2^{-k}) \quad (k = 1, 2, \cdots) \\ &= -A_2 \quad \text{otherwise on} \quad (0, \pi), \end{aligned} \tag{7}$$

where the constant A_2 is taken as the mean value of f_2 vanishes, then $f_2 \in M^p$, but not in L_0^p. Thus Proposition 1 is proved.

Furthermore, f_2 does not belong to any $L^{p''}$ $(1 < p'' < p)$, and then

$$L^{p''} \not\supset M^p \quad \text{for any } p'', \quad 1 < p'' < p.$$

3. Proof of Theorem 1. Hardy [1] has proved that

$$F^*(x) = \int_x^\pi f(t) \cot \frac{1}{2} t\, dt \sim \sum_{n=1}^\infty A_n^* \cos nx, \tag{8}$$

$$A_n^* = \frac{1}{n} \sum_{k=1}^{n-1} a_k + \frac{1}{2n} a_n. \tag{9}$$

Since

$$\sum_{n=1}^\infty n^{-1} a_n \cos nx \in L^{p'} \quad \text{for any } p' > 1, \tag{10}$$

it is sufficient to show that $F^* \in L^p \cap M^p$ when $f \in M^p$. By Minkowski's inequality,

$$\left(\int_0^\pi |F^*(x)|^p dx\right)^{1/p} \leqq A \int_0^\pi |f(u)|\, u^{-1/q} du$$

which is finite by $f \in M^p$. Hence $F^* \in L^p$. Further

$$\int_0^\pi |F^*(x)|\, x^{-1/q} dx \leqq A \int_0^\pi |f(u)|\, u^{-1/q} du.$$

Then $F^* \in M^p$ and the theorem is proved.

4. Proof of Proposition 3. It is sufficient to prove that there is a function $f \in L^p$ such that F does not belong to M^p. The function f_1, defined by (6), belongs to L^p. We define F_1^* by (8), using f_1 instead of f, then

$$\int_0^\pi F_1^*(x) x^{-1/q} dx \geqq A \int_0^\pi x^{-1/q} dx \int_x^\pi t^{-1-1/p} \left(\log \frac{2\pi}{t}\right)^{-1} dt = \infty.$$

Hence F_1^* does not belong to M^p and then, by (10), F_1 does not also. Thus f_1 is a function satisfying the required condition.

5. Proof of Proposition 4. We put $t_n = \frac{1}{2}\left(\frac{1}{\log n} + \frac{1}{\log (n+1)}\right)$ $(n=2, 3, \cdots)$ and consider the even and periodic function f_3 defined by

$$\begin{aligned} f_3(t) &= (\log n)^{1/p} \quad \text{in} \quad (1/\log (n+1),\, t_n) \\ &= -(\log n)^{1/p} \quad \text{in} \quad (t_n,\, 1/\log n) \qquad (n=2, 3, \cdots) \\ &= 0 \quad \text{in} \quad (1/\log 2,\, \pi). \end{aligned} \tag{11}$$

Then f_3 is evidently integrable and $\int_0^\pi |f_3(t)|\, t^{-1/q} dt = \infty$, that is, f_3 does not belong to M^p. But

$$\int_0^\pi t^{-1/q} dt \left| \int_t^\pi f_3(u) \cot \frac{1}{2} u\, du \right| < \infty, \quad \int_0^\pi \left| \int_t^\pi f_3(u) \cot \frac{1}{2} u\, du \right|^p dt < \infty,$$

and then F_3^* defined by f_3, belongs to $L^p \cap M^p$.

6. Proof of Theorem 3. We have

$$\int_\varepsilon^\pi F^*(x) x^{-1/q} dx = A \int_\varepsilon^\pi f(t) \cot \frac{1}{2} t \cdot t^{1/p} dt - A \varepsilon^{1/p} \int_\varepsilon^\pi f(t) \cot \frac{1}{2} t\, dt.$$

Theorem is proved when

$$\lim_{\varepsilon \to 0} \varepsilon^{1/p} \int_\varepsilon^\pi f(t) t^{-1} dt = 0. \tag{12}$$

Since $f \in M^p$, there is an $\eta > 0$, for any $\delta > 0$, such that

$$\left| \int_\varepsilon^{\varepsilon'} f(t) t^{-1/q} dt \right| < \delta \quad \text{for any} \quad \varepsilon < \varepsilon' < \eta. \tag{13}$$

By the mean value theorem and (13)

$$\limsup_{\varepsilon \to 0} \left| \varepsilon^{1/p} \int_\varepsilon^\pi f(t) t^{-1} dt \right| \leqq \limsup_{\varepsilon \to 0} \left| \int_\varepsilon^{\varepsilon'} f(t) t^{-1/q}\, dt \right| \leqq \delta.$$

Since δ is arbitrary, we get the required relation (12).

7. Proof of Proposition 6. We define the even and periodic function f_4 by the equations

$$\begin{aligned} f_4(t) &= (-1)^n (\log k)^{1/p} - A_4 \quad \text{on} \quad (1/\log k,\, 1/\log (k-1)) \\ &\qquad \text{for} \quad 2^n < k \leqq 2^{n+1} \quad (n=2, 3, \cdots) \\ &= -A_4 \quad \text{on} \quad (1/2 \log 2,\, \pi) \end{aligned} \tag{14}$$

where the constant A_4 is taken as the mean value of f_4 vanishes. Then

f_4 is integrable and belongs to N^p. If we define F_4^* by (8) using f_4, instead of f, then F_4^* does not belong to L^p.

8. Proof of Proposition 7. We define the even and periodic function f_5 by the equations

$$
\begin{aligned}
f_5(t) &= 2^k k^{-2} \quad \text{on} \quad (n_k^{-1}, n_k^{-1}+2^{-k}) \\
&= -2^k k^{-2} \quad \text{on} \quad (n_k^{-1}+2^{-k}, n_k^{-1}+2\cdot 2^{-k}) \quad (k=2,3,\cdots) \\
&= 0, \quad \text{otherwise on} \quad (0,\pi)
\end{aligned}
\tag{15}
$$

where $n_k = k^{2q}$ $(k=2,3,\cdots)$, then f_5 is integrable, but does not belong to N^p and F_5^*, defined by f_5, belongs to N^p. The function f_5 has the required property.

9. Proof of Theorem 4. By the formal calculation,

$$
G(t) \sim -\frac{1}{2}\sum_{k=1}^{\infty}\frac{a_k}{k} + \frac{1}{2}\cot\frac{1}{2}t\sum_{k=1}^{\infty}\frac{a_k}{k}\sin kt + \sum_{k=1}^{\infty}\frac{a_k}{2k}\cos kt.
\tag{16}
$$

If we denote by $H(t)$ the last term of (16), then H belongs to any L^p $(p>1)$ by (10). The term before the last of (16) is

$$
K(t) = \frac{1}{2}\cot\frac{1}{2}t\int_0^t f(u)du
\tag{17}
$$

which is integrable. We shall now show that the function $H(t)+K(t)$ has the same Fourier coefficients as $G(t)$, except for the constant term. The n-th Fourier coefficient of K is

$$
\frac{2}{\pi}\int_0^\pi \frac{1}{2}\cot\frac{1}{2}t\cos nt\,dt\int_0^t f(u)du = \frac{2}{\pi}\int_0^\pi f(u)du\int_u^\pi \frac{1}{2}\cot\frac{1}{2}t\cos nt\,dt
$$

where

$$
\int_u^\pi \frac{1}{2}\cot\frac{1}{2}t\cos nt\,dt = -\log\left(\sin\frac{1}{2}u\right) - \int_u^\pi \tilde{D}_n^*(t)dt,
$$

$\tilde{D}_n^*$ being the n-th modified conjugate Dirichlet kernel [7]. Thus we have, by elementary estimation,

$$
\frac{2}{\pi}\int_0^\pi f(u)du\int_u^\pi \frac{1}{2}\cot\frac{1}{2}t\cos nt\,dt = \frac{2}{\pi}\int_0^\pi f(u)\left(\sum_{k=n}^{\infty}{}^{*}\, k^{-1}\cos ku\right)du,
\tag{18}
$$

where $\sum^*$ denotes that the first term is halfed in the summation. Since $f(u)\log\frac{2\pi}{u}$ is integrable by the assumption and the series $\left(\log\frac{2\pi}{u}\right)^{-1}\sum_{k=1}^{\infty}k^{-1}\cos ku$ is boundedly convergent, we can interchange the order of summation and integration on the right side of (18). Combining this with the n-th Fourier coefficient of H, we get the required result.

Therefore, in order to prove the theorem, it is enough to show that $K \in L^p \cap M^p$, where K is defined by (17). Now

$$
\int_0^\pi \frac{dt}{t^{1+1/q}}\int_0^t |f(u)|du = \int_0^\pi |f(u)|du\int_u^\pi \frac{dt}{t^{1+1/q}} \leq A\int_0^\pi |f(u)|u^{-1/q}du,
$$

that is, $f \in M^p$. On the other hand, by Minkowski's inequality,

$$\left(\int_0^\pi \left| t^{-1}\int_0^t f(u)du\right|^p dt\right)^{1/p} \leqq \int_0^\pi du\left(\int_u^\pi |f(u)|^p t^{-p} dt\right)^{1/p} = A\int_0^\pi |f(u)|\, u^{-1/q} du.$$

Therefore $G \in L^p$. Thus the theorem is proved.

10. Proof of Propositions 5 and 9. The function f_3 defined by (11) is integrable, but does not belong to both L^p and M^p. $t^{-1}\int_0^t f_3(u)du$ is integrable and then G_3, defined by f_3, is equal to $H_3 + K_3$, except for addition of some constant. Since

$$\int_0^\pi t^{-1/q} dt \left| t^{-1}\int_0^t f_3(u)du\right| < \infty \quad \text{and} \quad \int_0^\pi \left| t^{-1}\int_0^t f_3(u)du\right|^p dt < \infty,$$

we get $G_3 \in L^p \cap M^p$. Evidently $G_3 \in O^p$. Thus f_3 gives the solution of Proposition 9. Proposition 5 is proved using the same function f_3.

11. Proof of Proposition 10. We shall define the even and periodic function f_6 by the equations

$$\begin{aligned} f_6(t) &= h_k \quad \text{on} \quad (n_k^{-1}, n_k^{-1} + m_k^{-1}) \\ &= -h_k \quad \text{on} \quad (n_k^{-1} + m_k^{-1}, n_k^{-1} + 2m_k^{-1}) \quad (k = 1, 2, \cdots) \\ &= 0 \quad \text{otherwise in} \quad (0, \pi), \end{aligned}$$

where $h_k = k^q (\log k)^{q-1}/(\log\log k)^2$, $m_4 = 4k^{q+1}(\log k)^q$ and $n_k = k^q (\log k)^q$. Then we can see that f_6 is integrable, $f_6 \in N^p$ and G_6, defined by f_6, does not belong to N^p, using that $f_6(u)\log\dfrac{2\pi}{u}$ is integrable.

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