147. General Theory of Mappings

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In his paper [1], J. R. Büchi considered the notion of functions on a set. Some of his results are true for the both set theories in the senses of G. Cantor and S. Leśniewski. In this paper, we concern with a theory of functions on a set in the sense of G. Cantor.

Let E, E' be two given sets, f a function from 2^E to $2^{E'}$, where 2^E, $2^{E'}$ denote the sets of all subsets of E, E' respectively.

J. R. Büchi [1] introduced a notion of *a pair of functions* $(f, \bar{f})$ as follows: f and $\bar{f}$ are a pair of functions, if, for any function f, there is a function $\bar{f}$ from $2^{E'}$ to 2^E such that $A' \cap f(A)=0$ implies $\bar{f}(A') \cap A=0$, where $A \in 2^E$, $A' \in 2^{E'}$. J. R. Büchi obtained some important properties on $(f, \bar{f})$ (see [1]). Among these properties, an important result is the representation of $\bar{f}$: $f(A')=\cap\{X \mid f(E-X) \subset E'-A'\}$.

If $(f, \bar{f})$ is a pair of functions, then for $\{A_\alpha\}$, $A_\alpha \subset E$, we have $f(\bigcup_\alpha A_\alpha)=\bigcup_\alpha f(A_\alpha)$ (see [1], p. 164). Hence f is a multiform mapping in the sense of Dubreil ([4]-[7]).

Further we have $\bar{f}(f(A)) \supset A$. To prove it, take an element x of A. Suppose that $\bar{f}(f(A)) \cap x=\phi$, then $f(A) \cap f(x)=0$, which contradicts to $f(x) \subset f(A)$.

For the empty set ϕ and E, we have $\bar{f}(f(\phi))=\phi$, $\bar{f}(f(E))=E$. Therefore the family $\mathfrak{M}$ of all subsets A of E such that $\bar{f}(f(A))=A$ is not empty.

Let $A=\bigcup_\alpha A_\alpha$, $A_\alpha \in \mathfrak{M}$, then we

$$\bar{f}(f(A))=\bar{f}(f(\bigcup_\alpha A_\alpha))=\bar{f}(\bigcup_\alpha f(A_\alpha))=\bigcup_\alpha \bar{f}(f(A_\alpha))=\bigcup A_\alpha=A.$$

Let $B=\bigcap_\alpha A_\alpha$, $A_\alpha \in \mathfrak{M}$, then

$$\bar{f}(f(B))=\bar{f}(f(\bigcap_\alpha A_\alpha)) \subset \bar{f}(\bigcap_\alpha f(A_\alpha)) \subset \bigcap_\alpha \bar{f}(f(A_\alpha))=\bigcap A_\alpha=B.$$

On the other hand, $B \subset \bar{f}(f(B))$ for any subset B of E.

For any subset $A \in \mathfrak{M}$, $(E-A) \cap \bar{f}(f(A))=(E-A) \cap A=\phi$. Hence

$$f(E-A) \cap f(A)=\phi.$$

This implies $\bar{f}(f(E-A)) \cap A=\phi$, and we have $\bar{f}(f(E-A)) \subset E-A$. Therefore, we have the following

Theorem 1. *The family $\mathfrak{M}$ of all subsets A such that $f(f(A))$*

1) In this Note, we shall assume that $f(x) \neq 0$ for every $x \in E$.

$=A$ *is a set-field.*

Following P. Dubreil [6], a function f is called *semi-uniform*, if for any elements x, y of E, $f(x)\cap f(y)\neq\phi$ implies $f(x)=f(y)$.

Theorem 2. *The function* $\bar{f}$ *for a semi-uniform function* f *is semi-uniform.*

Proof. Let $\bar{f}(x')\cap\bar{f}(y')\neq\phi$ for two elements x', y' of E', then there is an element $u\in\bar{f}(x')$, $\bar{f}(y')$. Hence $x', y'\in f(u)$. Take any element z of $\bar{f}(x')$, then we have $x'\in f(z)$. Hence $f(u)\cap f(z)\neq\phi$. Since f is semi-uniform, $f(u)=f(z)$. Therefore $y'\in f(z)$, and we have $z\in\bar{f}(y')$. Hence $\bar{f}(x')\subset\bar{f}(y')$. Similarly $\bar{f}(y')\subset\bar{f}(x')$.

We denote $\bar{f}(f(A))$, $\bar{f}(f(\bar{f}(f(A))))$, $\cdots$ for a pair of functions $(f,\bar{f})$ by $(\bar{f}f)(A)$, $(\bar{f}f)^2(A)$, $\cdots$. For any subset A of E, we define $h(A)$ by

(1) $$\{y\mid y\in(\bar{f}f)^n(A)\quad\text{for some } n\}.$$

Then h is a mapping from 2^E to 2^E.

For any element x, suppose that $f(x)\neq\phi$, hence $f(x)\cap f(x)\neq\phi$. Therefore $x\in(\bar{f}f)(x)$. By repeating the same argument, we have $x\in(\bar{f}f)^n(x)$ and consequently $x\in h(x)$. Hence for any subset A of E, $A\subset h(A)$, i.e., h is a reflexive relation on E (see [1], p. 163).

For any subsets A, B, suppose that $h(A)\cap B\neq\phi$. Then there is an element x such that $x\in h(A)\cap B$. Hence for some n and $y\in A$, $x\in(\bar{f}f)^n(y)$. Therefore $y\in(\bar{f}f)^n(x)$. This means $A\cap h(B)\neq\phi$. Hence we have $h=\bar{h}$, i.e., h is a symmetric relation (see [1], p. 163).

By the definition of $h(x)$, we have $h(h(A))\subset h(A)$, i.e., h is a transitive relation (see [1], p. 163). Therefore h is an equivalence relation on E.

Theorem 3. *The function* h *defined by* (1) *gives an equivalence relation on* E, *where* $f(x)\neq 0$ *for every* $x\in E$.

As already mentioned, a function $f:2^E\to 2^E$ satisfying 1) $A\subset f(A)$, 2) $f(A)=f(A)$, and 3) $f(f(A))\subset f(A)$ for every subset A of E is called an *equivalence relation* on E.

Let f be an equivalence relation on E. If $f(x)\cap f(y)\neq\phi$, then we have $x\cap f(f(y))\neq\phi$, i.e., $x\in f(f(y))$. By 3), $x\in f(y)$. Hence $f(x)\subset f(f(y))\subset f(y)$. Similarly we have $f(y)\subset f(x)$. This shows $f(x)=f(y)$. Therefore we have the following

Theorem 4. *An equivalence relation is semi-uniform.*

Let f, g be two equivalence relation of a set E. A function $g*f:2^E\to 2^E$ is defined by

$$(g*f)(A)=\{y\mid y\in(gf)^n(A)\ \text{ for some } n=0,1,2,\cdots\}.$$

For $g*f$, we have $A\subset(g*f)(A)$, and $(g*f)((g*f)(A))\subset(g*f)(A)$ by the definition $g*f$. Let $y\in(gf)^n(A)$. By 1), $A\subset g(A)$. Hence $f(A)\subset f(g(A))$. Consequently we have $(gf)^n(A)\subset g(fg)^n(A)$. Therefore

$y \in g(fg)^n(A)$, and then

$$y \in f(y) \subset f(g(fg)^n(A)) = (fg)^{n+1}(A).$$

This shows that $g * f$ is symmetric.

Theorem 5. *If f, g are two equivalence relation, then $g * f$ $(= f * g)$ is an equivalence relation.*

Let f, g be two equivalence relations on a set E. If, for any three elements a, b, and x, $a \in f(x)$ and $b \in g(x)$ imply $a \in g(y)$, $b \in f(y)$ for some $y \in E$, f, and g is called to be *associable* (This notion and Theorem 6 are essentially due to [3]).

Theorem 6. *If h is associable to f and g, then h is associable to $g * f$.*

Proof. Let $a \in h(x)$ and $b \in (g * f)(x)$, then $b \in (gf)^n(x)$ for some n. Hence $f(x) \cap g(fg)^{n-1}(b) \neq \phi$. Then there is an element $c_1 \in f(x) \cap g(fg)^{n-1}(b)$. $c_1 \in f(x)$, $a \in h(x)$ imply that there is an element d such that

$$c_1 \in h(d), \qquad a \in f(d).$$

$c_1 \in g(fg)^{n-1}(b)$ implies $g(c_1) \cap (fg)^{n-1}(b) \neq \phi$, and we find an element c_2 such that

$$c_2 \in g(c_1), \qquad c_2 \in (fg)^{n-1}(b).$$

By $d \in h(c_1)$, $c_2 \in g(c_1)$, there is an element e_1 such that $d \in g(e_1)$, $c_2 \in h(e_1)$. Therefore, by $a \in f(d)$ and $d \in g(e_1)$, we have

1) $a \in f(g(e_1))$.

On the other hand, by $c_2 \in (fg)^{n-1}(b)$ we have $b \in (gf)^{n-1}(c_2)$. Hence, by $c_2 \in h(e_1)$,

2) $b \in (gf)^{n-1}(h(e_1))$.

By repeating this technique, we find some element e_n such that $a \in (fg)^n(e_n)$ and $b \in h(e_n)$. This shows $a \in (g * f)(y)$ and $b \in h(y)$ for some y.

A modern algebraic theory of equivalence relations is found in [2], [3]. These results can be treated by our function method.

For two functions $f, g : 2^E \to 2^F$, we define $f \leqslant g$ by $f(A) \subset g(A)$ for every subset A of E. $(f \cap g)(A)$ is defined by $f(A) \cap g(A)$, and $(f \cup g)(A)$ by $f(A) \cup g(A)$.

Let $(f, \bar{f})$, $(g, \bar{g})$ be two pairs of functions. Suppose that

$$A' \cap (f(A) \cup g(A)) = 0,$$

then we have

$$A' \cap f(A) = 0, \qquad A' \cap g(A) = 0$$

and $\bar{f}(A') \cap A = \bar{g}(A') \cap A = 0$. Therefore

$$(\bar{f}(A') \cup \bar{g}(A')) \cap A = 0,$$

which means that $(f \cup g, \bar{f} \cup \bar{g})$ is a pair of functions. Hence we have $\overline{f \cup g} = \bar{f} \cup \bar{g}$.

Next we shall prove $\overline{f \cap g} \leqslant \bar{f} \cap \bar{g}$. Let $x \in (\overline{f \cap g})(A)$, then

$(f \cap g)(x) \cap A = 0$. There is an element y such that $y \in (f \cap g)(x) \cap A$, i.e., $y \in (f \cap g)(x)$, $y \in A$. Hence $y \in f(x)$, $g(x)$ and then $x \in \bar{f}(y)$, $\bar{g}(y)$. Therefore $x \in \bar{f}(A) \cap \bar{g}(A)$.

We prove the following Dedekind relation (see J. Riquet [8]).

Let $(f, \bar{f})$, $(g, \bar{g})$, and $(h, \bar{h})$ be pairs of functions: $2^E \to 2^F$, $2^F \to 2^G$ and $2^E \to 2^G$ respectively. Then we have

3) $((gf) \cap h)(x) \subset (g \cap (h\bar{f}))(f \cap (\bar{g}h))(x)$.

Let $y \in ((gf) \cap h)(x)$, then $y \in g(f(x))$, $y \in h(x)$. Hence $\bar{g}(y) \cap f(x) \neq \phi$, and $\bar{g}(y) \subset \bar{g}(h(x))$. From $y \in h(x)$, we have $x \in \bar{h}(y)$, and $f(x) \subset f(\bar{h}(y))$. Therefore

$$f(x) \cap \bar{g}(h(x)) \cap \bar{g}(y) \cap f(\bar{h}(y)) \neq \phi.$$

Hence

$$(f \cap (\bar{g}h))(x) \cap (\bar{g} \cap (f\bar{h}))(y) \neq \phi.$$

This implies

$$\begin{aligned} y \in (\overline{\bar{g} \cap (f\bar{h})})(f \cap (\bar{g}h))(x) &\subset (\bar{\bar{g}} \cap (\overline{f\bar{h}}))(f \cap (\bar{g}h))(x) \\ &= (g \cap (h\bar{f}))(f \cap (\bar{g}h))(x). \end{aligned}$$

Therefore we have the Dedekind relation 3).

References

[1] J. R. Büchi: Die Boole'sche Partialordnung und die Paarung von Gefügen. Portugallae Math., **7**, 119–180 (1948).

[2] A. Châtelet: Algèbre des relations de congruence. Annales scient. École normal, **64**, 339–368 (1947).

[3] P. Dubreil et M. L. Dubreil-Jacotin: Thèorie algébrique des relations d' équivalence. Jour. de Math., **18**, 63–95 (1939).

[4] M. L. Dubreil-Jacotin: Quelques propriétés des applications multiformes. C. R. de Paris, **230**, 806–808 (1950).

[5] ——: Applications multiformes et relations d'équivalences. C. R. de Paris, **203**, 906–908 (1950).

[6] P. Dubreil: Relations binaires et applications. C. R. de Paris, 1028–1030 (1950).

[7] ——: Comportment des relations binaires dans une application multiforme. C. R. de Paris, **230**, 1242–1244 (1950).

[8] J. Riquet: Relations binaires, fermetures, correspondences de Galois. Bull. Soc. Math. de France, **76**, 114–153 (1948).