142. Global Solution for an Initial Boundary Value Problem of a Quasilinear Hyperbolic System

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§1. **Introduction.** We consider the following system of equations

(1) $\qquad \partial v/\partial t-\partial u/\partial x=0, \qquad \partial u/\partial t+\partial(a^2/v)/\partial x=0,$

which is the simplest equation in gas dynamics (Lagrangean form in the isothermal case: $p=a^2/v$, a is a constant>0), where v is the specific volume, u is the speed of the gas.

Here we consider the Cauchy problem in $t\geq 0$, $-\infty<x<+\infty$ for (1) with the initial values

(2) $\qquad v(0, x)=v_0(x), \ u(0, x)=u_0(x)$ for $-\infty<x<+\infty$

and also the piston problem (an initial boundary value problem) in $t\geq 0$, $x\geq 0$ for (1) with the boundary values

(3) $\qquad v(0, x)=v_0(x), \ u(0, x)=u_0(x)$ for $x\geq 0$,

$\qquad u(t, 0)=u_1(t)$ for $t\geq 0$,

where $v_0(x)$, $u_0(x)$, $u_1(t)$ are bounded functions with locally bounded variation and $v_0(x)\geq\delta=$constant>0.

We see that the Cauchy problem (1), (2) and the piston problem (1), (3) have generalized solutions in the large. We use the Glimm's (or slightly modified) difference scheme [2] for the proof of the existence theorems.

There are many articles [1]-[8] which treat the existence theorem of the solution in the large for the initial value problem of the quasilinear hyperbolic system of equations, where the system is more general than in this paper, but the initial value is more restricted.

§2. **Cauchy problem.** Here we consider the Cauchy problem (1), (2). The definition of the generalized solution v, u of the Cauchy problem (1), (2) is the following: $v(t, x)$, $u(t, x)$ are bounded measurable functions and satisfy the integral identity

$$\iint_{t>0}(v\cdot f_t-u\cdot f_x)dtdx+\int_{t=0}v_0(x)f(0, x)dx=0$$

$$\iint_{t>0}(u\cdot g_t+(a^2/v)\cdot g_x)dtdx+\int_{t=0}u_0(x)g(0, x)dx=0$$

for any continuously differentiable functions f, g with compact support.

The system (1) is hyperbolic in $v>0$ and has the characteristics, the Riemann invariants and the nonlinearity as follows:

$$\lambda = -(a/v),\ r = u + a\log v;$$
$$\mu = a/v,\ s = u - a\log v;$$
$$\partial\lambda/\partial r = \partial\mu/\partial s = (1/2a)\exp\{-(r-s)/2a\} > 0.$$

Concerning the Riemann problem (1) with the initial values

$$(4)\qquad v_0(x) = \begin{cases} v_- \\ v_+ \end{cases},\quad u_0(x) = \begin{cases} u_- & x<0 \\ u_+ & x>0 \end{cases},$$

where $v_\mp$, $u_\mp$ are constants, $v_\mp > 0$,
we have the following lemma

Lemma 1. *Riemann problem* (1), (4) *has a generalised solution* $v(t, x)$, $u(t, x)$ *which is piecewise continuous and piecewise smooth, and also satisfies the following a priori estimate.*

$$(5)\qquad r(v(t, x), u(t, x)) \geq r_0,\ s(v(t, x), u(t, x)) \leq s_0,$$

where $r_0 = \min\{r(v_-, u_-), r(v_+, u_+)\}$, $s_0 = \max\{s(v_\mp, u_\mp)\}$.

Lemma 2. *The shock curve has the same figure in* (r, s) *plane, i.e., the first shock arising form* r_0, s_0 *is represented by*

$$(6.1)\qquad s - s_0 = f(r - r_0)\quad \text{for}\quad r \leq r_0,$$

the second shock is

$$(6.2)\qquad r_0 - r = f(s_0 - s)\quad \text{for}\quad s \leq s_0,$$

where $f(r)$ *is independent of* r_0, s_0 *and an odd function in* r *and* $0 \leq f'(r) \leq 1$.

Now we consider the Glimm's difference scheme [2]. We approximate the initial values (2) by the step functions

$$(7)\qquad v^l(0, x) = v_0(ml),\ u^l(0, x) = u_0(ml)$$
$$\text{for } (m-1)l < x < (m+1)l,\ \forall l > 0,\ m: \text{even},$$

and define

$$r_0 = \inf_{-\infty<x<+\infty} r(v_0(x), u_0(x)),\ s_0 = \sup_{-\infty<x<+\infty} s(v_0(x), u_0(x)),$$
$$(8)\qquad h/l = \exp\{(r_0 - s_0)/2a\}/a,$$
$$(9)\qquad \begin{aligned} &Y = \{(m, n): m, n \text{ integers, } m+n \text{ is even and } n \geq 1\} \\ &A = \prod_{(m,n)\in Y} [((m-1)l, (m+1)l) \times \{nh\}], \end{aligned}$$

where each factor is a horizontal line segment in the plane. We choose a point $a = \{a_{mn}\} \in A$ at random, and additionally we put $a_{m0} = ml$.

Suppose that our difference approximation $v^l = v^l(t, x)$, $u^l = u^l(t, x)$ has been defined for $(t, x) = a_{m-1,n-1}$ and $a_{m+1,n-1}$. Let v, u be the solution of the Riemann problem (1) for $t > (n-1)h$ with initial values

$$v((n-1)h, x) = \begin{cases} v^l(a_{m-1,n-1}) \\ v^l(a_{m+1,n-1}) \end{cases},\quad u((n-1)h, x) = \begin{cases} u^l(a_{m-1,n-1}) \\ u^l(a_{m+1,n-1}) \end{cases}$$

for $x < ml$, $ml < x$ respectively. Let

$$v^l(a_{mn}) = v(a_{mn}),\ u^l(a_{mn}) = u(a_{mn})$$

and

$$v^l = v,\ u^l = u\ \text{ for }\ (m-1)l \leq x \leq (m+1)l,\ (n-1)h \leq t < nh,$$

then v^l, u^l is a generalized solution in $(n-1)h \leq t < nh$ because

$$v^l(t, x) = v^l(a_{m\mp1,n-1}),\quad u^l(t, x) = u^l(a_{m\mp1,n-1})$$

for x near the boundary $(m\mp1)l$ by Lemma 1 and the definition of h in (8). Thus we have the difference approximation $v^l(t, x)$, $u^l(t, x)$ for $-\infty < x < +\infty$, $0 \le t \le T$, $\forall T > 0$, $\forall l > 0$.

In order to obtain the uniform boundedness of v, u in $0 \le t \le T$, $|x| \le X$ and the uniform boundedness of its total variation with respect to x for $|x| \le X$ on $t = \text{const.} > 0$ and the continuity

(10) $$\int_{-X}^{X} |v^l(t_2, x) - v^l(t_1, x)| + |u^l(t_2, x) - u^l(t_1, x)|\, dx \le C \cdot |t_2 - t_1| \quad \text{for } \forall X > 0$$

which assure [2] that some subsequence v^j, u^j of v^l, u^l converges in $L^1(loc)$ as $j \to 0$ and the limit is a generalized solution of (1), (2), we define the following functional on J_i, $i = 1, 2$, where J_1 (or J_2) denote the line-segments joining the mesh points $a_{m-1,n}$, $a_{m,n-1}$, $a_{m+1,n}$, (or $a_{m-1,n}$, $a_{m,n+1}$, $a_{m+1,n}$),

(11) $$F(J_i) = \Sigma_{J_i}(\Delta r + \Delta s),$$

where Δr (or Δs) is the variation of the Riemann invariant r (or s) in the first (or second) shock wave and Σ_{J_i} is the sum of them for all shocks on J_i.

Lemma 3.

(12) $$F(J)_2 \le F(J_1).$$

By means of this lemma we may have the desired estimate. Since the approximate solution has the finiteness of its dependence domain because of $|\lambda| = |\mu| = a/v \le a \cdot \exp\left(-\frac{r_0 - s_0}{2a}\right)$ and the choice of h in (8), when we consider the approximate solution on $t = t_0 \ge 0$, $-X \le x \le X$, we may change it for large $|x|$ i.e., $v^l(0, x) = v_0(-X - Ct_0)$, $u^l(0, x) = u_0(-X - Ct_0)$ for $|x| \ge X + Ct_0$. Let $\tilde{v}^l$, $\tilde{u}^l$ be the changed approximate solution.

Following Glimm let I-curve J be any curve consisting of line-segments joining the mesh points $a_{m,n}$, $a_{m+1,n+1}$ or $a_{m,n}$, $a_{m+1,n-1}$, on which the mesh index m increases monotonically. If (12) holds, we have

(13) $$F(J) \le F(0) \qquad \text{for } \tilde{v}^l, \tilde{u}^l \text{ and any } I\text{-curve } J,$$

where 0 is the unique I-curve which passes through the mesh points on $t = 0$ and $t = h$, because of the succesive application of (12). Since Riemann invariants r, s are decreasing in shock waves and increasing in rarefication waves and $\Delta r + \Delta s < 2\Delta r$ on the first shock, $\Delta r + \Delta s < 2\Delta s$ on the second shock by Lemma 2, we obtain the following

$$\underset{J}{total\ var.}\ \{\tilde{r}^l, \tilde{s}^l\} \le 2 \underset{\text{shocks on } J}{total\ var.}\ \{\tilde{r}^l, \tilde{s}^l\} \le 4F(J) \le 4F(0),$$

from this we have the uniform boundedness or $\tilde{r}^l = r(\tilde{v}^l, \tilde{u}^l)$, $\tilde{s}^l = s(\tilde{v}^l, \tilde{u}^l)$ and so by I-curve J which coincides I-curve 0 for $|X| \ge X + Ct_0$

$$\begin{aligned}(14)\qquad & \underset{t=t_0,|x|\leq X}{total\ var.}\{v^l, u^l\}\leq \underset{J \text{ for } |x|\leq X}{total\ var.}\{\tilde{v}^l, \tilde{u}^l\} \\ & \leq C \underset{J \text{ for } |x|\leq X}{total\ var.}\{\tilde{r}^l, \tilde{s}^l\}\leq C \underset{t=0,|x|\leq X+Ct_0}{total\ var.}\{\tilde{r}^l, \tilde{s}^l\} \\ & \leq C \underset{t=0,|x|\leq X+Ct_0}{total\ var.}\{v^l, u^l\},\end{aligned}$$

$$\begin{aligned}(15)\qquad & v^l(t, x)\leq v_0(-X-Ct)+\underset{J}{total\ var.}\{\tilde{v}^l\}\leq C \text{ for } |x|\leq X \\ & |u^l(t, x)|\leq |u_0(-X-Ct)|+\underset{J}{total\ var.}\{\tilde{u}^l\}\leq C \text{ for } |x|\leq X.\end{aligned}$$

From these we have also (10).

Theorem 1. *The Cauchy problem* (1), (2) *has a generalized solution in* $t\geq 0$, *which is locally bounded and has locally bounded total variation in* x *on* $t=\text{const.}\geq 0$.

§ 3. Piston problem. Here we consider the initial boundary value problem (1), (3). The definition of the generalized solution v, u of (1), (3) is the following: $v(t, x)$, $u(t, x)$ are bounded measurable functions in $t\geq 0$, $x\geq 0$ and satisfy the following integral identity.

$$(16)\qquad \iint_{t>0,x>0}(v\cdot f_t-u\cdot f_x)dtdx+\int_0^\infty v_0(x)f(0, x)dx+\int_0^\infty u_1(t)f(t, 0)dt=0$$

for any $f\in \mathring{C}^1$,

$$(17)\qquad \iint_{t>0,x>0}\left(u\cdot g_t+\frac{a^2}{v}\cdot g_x\right)dtdx+\int_0^\infty u_0(x)g(0, x)dx=0$$

for any $g\in \mathring{C}^1$ and $g(t, 0)=0$.

We see that the problem (1), (3) has a generalized solution in the large. We use the modified Glimm's difference scheme for the proof of this existence theorem.

First we consider in $t\geq 0$, $x\geq 0$ the initial boundary value problem for (1) with the following data.

$$\begin{aligned}(18)\qquad & v(0, x)=v_+,\ u(0, x)=u_+ && \text{for } x>0, \\ & u(t, 0)=u_- && \text{for } t>0,\end{aligned}$$

where v_+, $u_\pm$ are const. s $v_+>0$.

Lemma 4. The problem (1), (18) has a generalized solution in $t\geq 0$, $x\geq 0$, which is piecewise continuous, piecewise smooth and also satisfies the following estimate.

$$(19)\qquad \begin{cases} r(t, x)\equiv r(v(t, x), u(t, x))\geq r(v_+, u_+)\equiv r_+, \\ s(t, x)\equiv s(v(t, x), u(t, x))\leq \max\{s_+\equiv s(v_+, u_+), 2u_--r_+\}, \\ \Delta s\leq C|u_+-u_-|\equiv C\cdot\Delta u, \end{cases}$$

where Δs is the variation of Riemann invariant s in the second shock in the solution, C is a const. independent of the data. Let

$$\begin{aligned} & r_0=\inf_{x\geq 0} r(v_0(x), u_0(x)), \\ & s_0=\max\{\sup_{x\geq 0} s(v_0(x), u_0(x)), 2\sup_{0\leq t\leq T} u_1(t)-r_0\}.\end{aligned}$$

Now we define the modified Glimm's difference scheme as follows:

$$(20)\qquad \begin{cases} Y=\{(m, n): m=0, 2, 4, \cdots, n=1, 2, 3, \cdots\}, \\ A=\prod_{(m,n)\in Y}[(ml, (m+2)l)\times\{nh\}], \end{cases}$$

where $h/l=\exp\{(r_0-s_0)/2a\}/a$. Let the mesh points $a=\{a_{mn}\}$ be chosen arbitrarily in A. Lemma 1 and Lemma 4 assure that the modified Glimm's scheme is defined for $0\leq t\leq {}^{\forall}T$.

Let i-curve i_m^{n-} (or i_m^{n+}), $m\geq 2$, be the smooth space-like line joining the mesh points $a_{m-2,n}$, a_{mn} which lies in $(n-1)h<t\leq nh$ (or $nh\leq t<(n+1)h$) and does not pass through the point (nh, ml); $i_0^{n\pm}$ is the straight line segment joining the points $(nh\pm h/2, 0)$ and $a_{0,n}$. I-curve J is composed of i-curves $i_m^{n\pm}$, $m\geq 0$ and the straight line segments joining the mesh points a_{mn}, $a_{m+2,n-1}$ or a_{mn}, $a_{m+2,n+1}$ on which the mesh index m increases monotonically.

Lemma 5.

$$(21)\qquad \begin{aligned} &F(i_m^{n+})\leq F(i_m^{n-}) &&\text{for } m\geq 2, \\ &F(i_0^{n+})\leq F(i_0^{n-})+C\cdot \Delta u_1 &&\text{for } m=0, \end{aligned}$$

where $F(\cdot)$ is the same as (11) and $\Delta u_1=|u_1(nh+h/2)-u_1(nh-h/2)|$.

From this lemma we have analogously the desired bounds for the approximate solution $v^l(t, x)$, $u^l(t, x)$, that is, u^l has the uniform boundedness of the locally total variation and u^l is uniformly locally bounded, then v^l has the analogous estimate.

Theorem 2. *The piston problem* (1), (3) *has a generalised solution in the large, which is locally bounded and has locally bounded variation with respect to* $x\geq 0$ *on* $t=$const.≥ 0 *and is continuous as vector valued function in* $L^1(0\leq x\leq {}^{\forall}X)$ *from* $t\geq 0$.

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