141. The Characters of Some Induced Representations of Semisimple Lie Groups

By Takeshi HIRAI
Department of Mathematics, Kyoto University

(Comm. by Kinjirô KUNUGI, M. J. A., Sept. 12, 1968)

Introduction. Let G be a simply connected semisimple Lie group. Let $\mathfrak{g}_0$ be its Lie algebra and let $\mathfrak{g}_0=\mathfrak{k}_0+\mathfrak{p}_0$ be a Cartan decomposition of $\mathfrak{g}_0$, where $\mathfrak{k}_0$ is a maximal compact subalgebra of $\mathfrak{g}_0$. Let us fix arbitrarily a maximal abelian subalgebra $\mathfrak{h}_0^-$ of $\mathfrak{p}_0$. Let $\mathfrak{g}$ and $\mathfrak{h}^-$ be the complexifications of $\mathfrak{g}_0$ and $\mathfrak{h}_0^-$ respectively. Introduce a lexicographic order in the set of all roots of $\mathfrak{g}$ with respect to $\mathfrak{h}^-$ and let Δ be the set of all positive roots of $\mathfrak{g}$.

Fix an element $\boldsymbol{h}_0\neq 0$ of $\mathfrak{h}_0^-$ and let Δ' be the set of all roots $\alpha\in\Delta$ zero at $\boldsymbol{h}_0$ and Δ'' the complement of Δ' in Δ. Let $\mathfrak{h}_0'$ be the subalgebra of $\mathfrak{h}_0^-$ orthogonal to Δ'. Consider the centralizer S of $\mathfrak{h}_0'$ in G. Let S_1 be a subgroup of S and let $s\rightarrow L_s$ $(s\in S_1)$ be a representation of S_1 by bounded operaters on a Hilbert space E. If S_1 and L fulfill some conditions, we can construct canonically a representation of G on a certain Hilbert space, starting from L (see §1). After F. Bruhat [1] we call it induced representation of L and denote it by T^L. He has studied in [1] a criterion of the irreducibility of T^L, when L is of finite-dimensional. Our present purpose is (1) to obtain a sufficient condition on S_1 and L for the existence of the characters of both L and T^L, and (2) to express the character of T^L by that of L in the form of summation. This has been done in very special cases in [2], [3], and [4(b)].

§1. Induced representations. Let $\mathfrak{c}_0$ be the center of $\mathfrak{k}_0$ and put $\mathfrak{k}_0'=[\mathfrak{k}_0,\mathfrak{k}_0]$, then $\mathfrak{k}_0=\mathfrak{c}_0+\mathfrak{k}_0'$. For any $\alpha\in\Delta$, let $\mathfrak{g}_\alpha$ be the set of all elements x of $\mathfrak{g}$ which fulfill

$$[\boldsymbol{h}, x]=\alpha(\boldsymbol{h})x \quad (\boldsymbol{h}\in\mathfrak{h}^-).$$

Put $\mathfrak{n}=\sum_{\alpha\in\Delta}\mathfrak{g}_\alpha$, $\mathfrak{n}'=\sum_{\alpha\in\Delta''}\mathfrak{g}_\alpha$, $\mathfrak{n}_0=\mathfrak{n}\cap\mathfrak{g}_0$, and $\mathfrak{n}_0'=\mathfrak{n}'\cap\mathfrak{g}_0$. Then $\mathfrak{n}_0$ and $\mathfrak{n}_0'$ are subalgebras of $\mathfrak{g}_0$. Let K, H^-, D, K', N, and N' be the analytic subgroups of G corresponding to $\mathfrak{k}_0$, $\mathfrak{h}_0^-$, $\mathfrak{c}_0$, $\mathfrak{k}_0'$, $\mathfrak{n}_0$, and $\mathfrak{n}_0'$ respectively. Then $G=NH^-K$ is Iwasawa decomposition of G.

We assume that the subgroup S_1 fulfills that

$$S^0(D\cap Z)\subset S_1\subset S,$$

where S^0 is the connected component of the identity element of S and Z is the center of G. Moreover we assume on L that L_z is a scalar multiple of the identity operator 1_E on E for every $z \in D \cap Z$.

Now let us construct explicitely the representation T^L. Put $\Gamma_1 = N'S_1$, then Γ_1 is a closed subgroup of G and N' is normal in Γ_1. Put $\Xi_1 = K \cap S_1$, then $\Gamma_1 = NH^- \Xi_1$, and therefore $G = \Gamma_1 K$ (see [1]). If an element $g \in G$ has two expressions $g = \gamma u = \gamma' u'$ ($\gamma, \gamma' \in \Gamma_1$, $u, u' \in K$), there exists an element $\xi \in \Xi_1$ such that $\gamma' = \gamma \xi$, $u' = \xi^{-1} u$. For any $g \in G$, let $\boldsymbol{h}(g)$ be the unique element of $\mathfrak{h}_0^-$ such that $g \in N \exp(\boldsymbol{h}(g))K$ and define

$$\beta(g) = \exp\{\sum_{\alpha \in \Delta''} n_\alpha \alpha(\boldsymbol{h}(g))\}, \tag{1}$$

where $n_\alpha = \dim_C \mathfrak{g}_\alpha$. For any $g \in G$, let $\Gamma(g)$ be the unique element of $\mathfrak{c}_0$ such that $g \in NH \exp(\Gamma(g))K'$. Then there exists uniquely two real-valued linear functions ν and ν' on $\mathfrak{c}_0$ such that

$$L_z = \exp\{(\nu + \sqrt{-1}\,\nu')(\Gamma(z))\} 1_E \tag{2}$$

for every $z \in D \cap Z$. Denote by K^* the factor group $K/D \cap Z$ and let du^* ($u^* \in K^*$) be the normalized Haar measure on K^* such that $\int_{K^*} du^* = 1$.

Let $\mathfrak{A}$ be the set of all functions on K with values in E which fulfill the following three conditions:

(a) $f(\xi u) = L_\xi f(u)$ for any $u \in K$, $\xi \in \Xi_1$;

(b) $(f(u), a)$ is measurable on K for every $a \in E$, where (,) denotes the inner product of E;

(c) $\int_{K^*} (f(u), f(u)) e^{-2\nu(\Gamma(u))} du^* < \infty$.

Note that the function $\varphi(u) = (f(u), f(u)) e^{-2\nu(\Gamma(u))}$ satisfies that $\varphi(zu) = \varphi(u)$ ($u \in K$, $z \in D \cap Z$) and therefore φ may be considered as a function on K^* and then φ is measurable on K^* from (b). Introduce in $\mathfrak{A}$ the inner product

$$(f_1, f_2) = \int_{K^*} (f_1(u), f_2(u)) e^{-2\nu(\Gamma(u))} du^* \tag{3}$$

and identify two functions f_1 and f_2 if $f_1(u) = f_2(u)$ for almost all u. Then we obtain a Hilbert space $L_2^L(K)$.

For every $g \in G$, define T_g^L as follows. First extend the representation L of S_1 to that of $\Gamma_1 = N'S_1$ by putting $L_n = 1_E$ for every $n \in N'$. Let $ug = \gamma u'$ ($\gamma \in \Gamma_1$, $u \in K$). For any $f \in L_2^L(K)$, put

$$T_g^L f(u) = \beta^{\frac{1}{2}}(\gamma) L_\gamma f(u'). \tag{4}$$

Then it is not difficult to prove that T_g^L is well-defined for any $g \in G$ and $g \to T_g^L$ defines actually a representation of G on $L_2^L(K)$. If L is unitary, T^L is also unitary. If L_1 and L_2 are topologically equivalent, T^{L_1} and T^{L_2} are also topologically equivalent.

§ 2. Existence of the characters. Let A be a bounded operator on a seperable Hilbert space $\mathcal{H}$. A is called summable if there exists a complete orthonormal system $a_1, a_2, a_3, \cdots$ of $\mathcal{H}$ such that

$$\sum_{i,j=1}^{\infty} |(Aa_i, a_j)| < \infty.$$

Let $\boldsymbol{\omega}$ be the set of all equivalent classes of finite-dimensional irreducible representations of Ξ_1. For any $\delta \in \boldsymbol{\omega}$, let $E(\delta)$ be the set of all vectors a in E such that $\{L_\xi a ; \xi \in \Xi_1\}$ spans a finite-dimensional vector space on which L_ξ ($\xi \in \Xi_1$) operate as a multiple of a representation of class δ. Denote by $C_0^\infty(S_1)$ the set of all functions on S_1 which are indefinitely differentiable and vanish outside some compact sets. Difine for $x \in C_0^\infty(S_1)$,

$$L_x = \int_{S_1} L_s x(s) ds,$$

where ds is a Haar measure on S_1. (ds is two-sided invariant, because S^0 is reductive and the index $[S_1 : S^0(D \cap Z)]$ is finite.) Analogously we define T_x^L for any $x \in C_0^\infty(G)$. We can prove the following lemma after Harish-Chandra [4(a)].

Lemma 1. *Suppose that L fulfills*

(L1) $$\dim E(\delta) \leqslant N d(\delta)^2$$

for any $\delta \in \omega$, where N is a constant independent of δ and $d(\delta)$ is the dimension of a representation of class δ. Then L_x is summable for any $x \in C_0^\infty(S_1)$ and the mapping $x \rightarrow Sp(L_x)$ is a distribution on S_1 in the sense of L. Schwartz (this distribution is called the character of L). Moreover T_x^L is summable for any $x \in C_0^\infty(G)$ and T^L has its character, a distribution on G.

A sufficient condition for that the character of L is essentially a locally summable function on S_1 can be deduced from the result of Harish-Chandra [4(c), Th. 2, p. 477]. Let us state it. Let $\boldsymbol{C}$ be the field of complex numbers. Consider the following conditions on S_1 and L.

(S) There exists a subgroup Z_1 of $S_1 \cap Z$ which has the following property: denote by $\underline{S}_1$ and $\underline{S}^0$ the factor groups S_1/Z_1 and S^0/Z_1 respectively and let $\underline{Z}_{S_1}$ be the center of $\underline{S}_1$, then $\underline{S}_1 = \underline{S}^0 \underline{Z}_{S_1}$.

(L2) $L_z = 1_E$, for every element $z \in Z_1$. And $L_z = \lambda(z) 1_E$ ($\lambda(z) \in \boldsymbol{C}$), for every element z of the inverse image of $\underline{Z}_{S_1}$ by the natural mapping $S_1 \rightarrow \underline{S}_1$.

(L3) Let $\mathfrak{s}_0$ be the Lie algebra of S_1 and $\mathfrak{s}$ its complexification. Let $\mathfrak{Z}_{\mathfrak{s}}$ be the center of the universal envelopping algebra of $\mathfrak{s}$. Then

$$L_z a = \alpha(z) a \qquad (z \in \mathfrak{Z}_{\mathfrak{s}}, \alpha(z) \in \boldsymbol{C})$$

for any vector a in E which is indefinitely differentiable under L.

Lemma 2. *If S_1 and L fulfill Conditions* (S), (L1), (L2), *and* (L3), *then the character τ of L is essentially a locally summable function on S_1 which is analytic on $S_1 \cap G'$.*

As will be seen in Theorem 2 in §3, in this case, the character π of T^L is also a locally summable function on G which is analytic on G', and is expressed by means of τ by means of a simple formula.

3. Expression of π by means of τ. Denote by $C_0(G)$ the set of all continuous functions on G which vanish outside some compact sets. Let $d\tilde{u}$ ($\tilde{u}=\Xi_1 u$) be the normalized invariant measure on the coset space $\tilde{K}_1=\Xi_1\backslash K$ such that $\int_{\tilde{K}_1} d\tilde{u}=1$. And let dn and ds be Haar measures on N' and S_1 respectively. Then we can normalize Haar measure dg on G in such a way that for any $x\in C_0(G)$,

$$\int_G x(g)dg=\int_{\tilde{K}_1} d\tilde{u}\int_{N'} dn\int_{S_1} x(nsu)\beta^{-1}(s)ds \quad (\tilde{u}=\Xi_1 u). \tag{6}$$

Theorem 1. *Under Condition* (L1) *on L in Lemma* 1, *the character π of T^L is expressed as follows: for any $x\in C_0^\infty(G)$,*

$$\int_G x(g)\pi(g)dg=Sp(T_x^L)$$
$$=\int_{\tilde{K}_1} d\tilde{u}\int_{N'} dn\int_{S_1} x(u^{-1}nsu)\beta^{-\frac{1}{2}}(s)\tau(s)ds, \tag{7}$$

where $\tilde{u}=\Xi_1 u$.

Now let us assume that S_1 and L fulfill the four conditions in Lemma 2. Then L may be considered as a representation of the factor group $\underline{S}_1$. Moreover since $T_z^L=1$ for every $z\in Z_1$, T^L may also be considered as a representation of the factor group $\underline{G}=G/Z_1$. Correspondingly, the characters τ and π may be considered as distributions on $\underline{S}_1$ and $\underline{G}$ respectively. For our purpose, there exists no essential difference for considering $\underline{G}$ and $\underline{S}_1$ instead of G and S_1. Therefore hereafter, for convenience, we consider $\underline{G}$ instead of G and denote $\underline{G}$, $\underline{S}_1$, $\underline{S}^0$, and $\underline{Z}_{S_1}$ simply by G, S_1, S^0, and Z_{S_1} respectively (S/Z_1 is denoted again by S and so on). The notations L, T^L, τ, and π etc. are preserved. Then the integral formula (7) remains valid. Since S_1 is the product of its center Z_{S_1} and its connected component S^0, some properties of connected reductive Lie groups are immediately translated on S_1.

Let $\mathfrak{h}_0^1, \mathfrak{h}_0^2, \cdots, \mathfrak{h}_0^k$ be a maximal set of Cartan subalgebras of $\mathfrak{s}_0$ which are not conjugate to each other under any inner automorphism of S_1. Then every $\mathfrak{h}_0^j$ is also a Cartan subalgebra of $\mathfrak{g}_0$. Let A^j (H^j resp.) be the Cartan subgroup of S_1 (G resp.) corresponding to $\mathfrak{h}_0^j$, that is, the centralizer of $\mathfrak{h}_0^j$ in S_1 (G resp.). Then clearly $H^j\cap S_1=A^j$ and $H^j\subset S$. Now let A_0^j (H_0^j resp.) be the center of A^j (H^j resp.) and put $Z^j=H_0^j\cap A_0^j$. Then the indices $p_j=[H_0^j:Z^j]$ and $q_j=[A_0^j:Z^j]$ are finite. If G is complex semisimple, $k=1$ S and H^1 are connected, and therefore $S_1=S, H^1=A^1=H_0^1=A_0^1$, and $p_1=q_1=1$. Let $\tilde{A}^j$ ($\tilde{H}^j$ resp.) be the normaliser of $\mathfrak{h}_0^j$ in S_1 (G resp.). Put $W_{A^j}=\tilde{A}^j/A_0^j$ and $W_{H^j}=\tilde{H}^j/H_0^j$.

Then W_{H^j} operates on H^j as $h \to h^\omega = ghg^{-1}$, where $\omega \in W_{H^j}$ and g is an element of the coset ω. Similarly W_{A^j} operates on A^j. The order w_j of W_{A^j} is finite. For an element $g \in G$, let $\mathrm{D}(g)$ be the coefficient of t^l in $\det(Ad(g) - 1_{\mathfrak{g}} + t1_{\mathfrak{g}})$, where $l = \mathrm{rank}\, G = \dim \mathfrak{h}_0^j$ and $1_{\mathfrak{g}}$ is the identity mapping on $\mathfrak{g}$. For $s \in S_1$, let $Ad_{\mathfrak{s}}(s)$ denote the restriction on $\mathfrak{s}$ of $Ad(s)$ and let $\mathrm{D}_{\mathfrak{s}}(s)$ be the coefficient of $t^{l-l'}$ in $\det(Ad_{\mathfrak{s}}(s) - 1_{\mathfrak{s}} + t1_{\mathfrak{s}})$ where $l' = \dim \mathfrak{h}_0'$.

Theorem 2. *If the four conditions in Lemma 2 are satisfied, we change the notations as is cited above. Then the character π is essentially a locally summable function on G which is expressed as follows: for any $g \in G'$,*

$$\pi(g) = |\mathrm{D}(g)|^{-\frac{1}{2}} \times \{\sum_{j=1}^{k} \frac{p_j}{q_j w_j} \sum_{\omega \in W_{H^j}, h_g^\omega \in A^j} |\mathrm{D}_{\mathfrak{s}}(h_g^\omega)|^{\frac{1}{2}} \tau(h_g^\omega)\}, \tag{8}$$

where h_g is an element of some H^j such that $g = g_0 h_g g_0^{-1}$ for some $g_0 \in G$, and the last summation runs over all $\omega \in W_{H^j}$ such that $h_g^\omega = (h_g)^\omega \in A^j$.

§ 4. Deduction of Theorem 2 from Theorem 1. Now we mention briefly to the proofs of Theorems 1 and 2. The proof of Theorem 1 is reduced essentially to a calculation of the trace of a certain integral operator, by using our special realization of the induced representation of L.

To deduce Theorem 2 from Theorem 1 under the four conditions in Lemma 2, the following integral formulas are essential. Take a Cartan subgroup H^j of G. For a subset A of H^j, denote by G_A the set of all elements of the form ghg^{-1} ($h \in A \cap G'$, $g \in G$). Let $g \to \bar{g}$ be the natural mapping of G onto $\bar{G} = H_0^j \backslash G$. Let $h \in H^j$ and $g \in G$. Denote $g^{-1}hg$ by $h^{\bar{g}}$ where $\bar{g} = H_0^j g \in \bar{G}$. Let dg, $d\bar{g}$, and dh be invariant measures on G, $\bar{G}$ and H^j respectively such that for any $x \in C_0(G)$,

$$\int_G x(g)dg = \int_{\bar{G}} d\bar{g} \int_{H_0^j} x(hg)dh \qquad (\bar{g} = H_0^j g). \tag{9}$$

Lemma 3. *Let A be an open subset of $H^j \cap G'$ and let f and φ be measurable functions on G_A and A respectively. If $f(h^{\bar{g}})\varphi(h)$ is integrable on $\bar{G} \times A$, then*

$$\int_{\bar{G}} d\bar{g} \int_A f(h^{\bar{g}})\varphi(h)dh = \int_{G_A} f(g)\{\sum_{\omega \in W_{H^j}, h_g^\omega \in A} \varphi(h_g^\omega)\} |\mathrm{D}(g)|^{-1} dg, \tag{10}$$

where h_g is an element of H^j such that $g = g_0 h_g g_0^{-1}$ for some $g_0 \in G$ and the summation runs over all ω for which $h_g^\omega \in A$ [see 4(c), p. 488, and 4(b), p. 508].

The analogous integral formula is valid for S_1 and A^j (recall that $S_1 = S^0 Z s_1$). From these formulas for $j = 1, 2, \cdots, k$, we obtain the following lemma. For every j $(1\leqslant j\leqslant k)$, let $d^j h$ $(h\in A^j)$ and $d^j\bar{s}$ $(\bar{s}=A_0^j s)$ be invariant measures on A^j and $\bar{S}^j=A_0^j\backslash S_1$ such that for any $x\in C_0(S_1)$,

$$\int_{S_1} x(s)ds=\int_{\bar{S}^j} d^j\bar{s}\int_{A_0^j} x(hs)d^j h. \tag{11}$$

Lemma 4. *If f is an integrable function on S_1, then*

$$\int_{S_1} f(s)ds=\sum_{j=1}^{k}\frac{1}{w_j}\int_{\bar{S}^j} d^j\bar{s}\int_{A^j} f(s^{-1}hs)\,|\mathrm{D}_{\mathfrak{s}}(h)|\,d^j h, \tag{12}$$

where $\bar{s}=A^j s$ for the j-th summand.

To deduce Theorem 2 from Theorem 1, first apply Lemma 4 to the integral (7) and next apply Lemma 3 to it, then we obtain the formula (8).

Note. Let G be the group of all real unimodular matrices of order n and take $S_1=S$. Then Condition (S) is generally not satisfied in this case. But for certain known representations of S, we can prove the analogous formula as that in Theorem 2. For the universal covering group of the above group G, we can find examples for which $p_j>1$ and $q_j>1$. Let G_1 be the group of all $n\times n$ matrices g such that det $g=\pm 1$. Then G_1 is not connected, but we can extend Theorem 2 to G_1 in certain cases.

References

[1] F. Bruhat: Sur les représentations induites des groupes de Lie. Bull. Soc. Math. France, **84**, 97–205 (1956).

[2] I. M. Gel'fand and M. I. Graev: Traces of unitary representations of real unimodular groups (in Russian). Doklady Acad. Nauk S. S. S. R., **100**, 1037–1040 (1955).

[3] M. I. Graev: Unitary representations of real simple Lie groups (in Russian). Trudy Mosk. Mat. Ob., **7**, 335–389 (1958).

[4] Harish-Chandra:

(a) Representations of semisimple Lie groups. III. Trans. Amer. Math. Soc., **75**, 234–253 (1954).

(b) The Plancherel formula for complex semisimple Lie groups. Ibid., **76**, 485–528 (1954).

(c) Invariant eigendistributions on semisimple Lie groups. Ibid., **119**, 457–508 (1965).