140. *Linear Set of the Second Category with Zero Capacity*

By Masanori KISHI and Mitsuru NAKAI
Mathematical Institute, Nagoya University

(Comm. by Kinjirô KUNUGI, M. J. A., Sept. 12, 1968)

1. In Cluster Set Theory various kinds of scales are used to describe the smallness of sets of exceptional characters (cf. Noshiro [2]), among which main ones are:

(a) Set theoretical one (cardinal numbers);
(b) Topological one (Baire category);
(c) Measure theoretic one (linear measure, e.g.);
(d) Potential theoretic one (logarithmic capacity, e.g.).

As for the relationship among them, excluding known or trivial ones we are here interested in those between (b) and (c), and (b) and (d). Existence of linear sets of the first category with positive or zero linear measure (or logarithmic capacity) can be seen on taking suitable generalized Cantor linear sets. Existence of linear sets of the second category with positive linear measure (or logarithmic capacity) is trivial. Therefore the problem is whether there exists linear set of the second category with zero linear measure (or logarithmic capacity). We shall show in 3 and 4 that such a set of zero logarithmic capacity (and hence of zero linear measure) exists.

2. Before proceeding to the construction in 3 and 4 we pause here, for the sake of completeness and comparison, to state the existence proof due to Professor Kiyoshi Noshiro (orally communicated to the authors) of the linear set of the second category with zero linear measure.

Let Γ be the unit circle and U the unit disk. Take a sequence $\{z_n\}_1^\infty$ of points $z_n \in U$ with $\sum_1^\infty(1-|z_n|)<\infty$ such that the totality of accumulation points of $\{z_n\}_1^\infty$ coincides with Γ. Let f be the Blaschke product whose zero set is $\{z_n\}_1^\infty$. Then the cluster set $C_U(f, z_0)$ contains zero for all $z_0 \in \Gamma$. On the other hand the angular cluster set $C_\Delta(f, z_0)$ for every Stolz angle Δ at z_0 consists of only one point with modulus 1 for almost all z_0 in Γ. Therefore the set $J(f)=\{z_0 \in \Gamma \mid C_\Delta(f, z_0) = C_U(f, z_0)$ for every $\Delta\}$ is of linear measure zero. It is known that $J(f)$ is the complement of a set of the first category in Γ and hence it is of the second category (Collingwood [1]). The existence proof of sets of the second category with zero linear measure is herewith complete.

3. We proceed to the construction anticipated in 1. Let I be the unit closed interval [0, 1]. Remove the interval $I_{1,1}^{(1)}$ of length $1-1/p_1^{(1)}$ centrally from I. Again remove congruent intervals $I_{2,1}^{(1)}$, $I_{2,2}^{(1)}$ of total length $(1/p_1^{(1)})(1-1/p_2^{(1)})$ centrally from two remaining intervals in $I-\overline{I_{1,1}^{(1)}}$. Repeating the same procedure we obtain the generalized Cantor set $E^{(0)}=E(p_1^{(1)}, p_2^{(1)}, \cdots, p_n^{(1)}, \cdots)$. Shrink and translate the newly taken Cantor set $E(p_1^{(2)}, p_2^{(2)}, \cdots, p_n^{(2)}, \cdots)$ so that it is placed on $\overline{I_{n_1,m_1}^{(1)}}(n_1=1, 2, \cdots ; m_1=1, 2, \cdots, 2^{n_1-1})$, which we shall denote by $E_{n_1,m_1}^{(1)}$. Denote by $I_{n_1,m_1;n_2,m_2}^{(2)}(n_2=1, 2, \cdots ; m_2=1, 2, \cdots, 2^{n_1-1})$ intervals removed to form the Cantor set $E_{n_1,m_1}^{(1)}$. Again shrink and translate $E(p_1^{(3)}, p_2^{(3)}, \cdots, p_n^{(3)}, \cdots)$ so as to be placed on $\overline{I_{n_1,m_1;n_2,m_2}^{(2)}}(n_\nu=1, 2, \cdots ; m_\nu=1, 2, \cdots, 2^{n_\nu-1}(\nu=1, 2))$, which we shall denote by $E_{n_1,m_1;n_2,m_2}^{(2)}$. Repeating this process we obtain the sequence $\{E_{n1,m_1;n_2,m_2;\cdots;n_k,m_k}^{(k)} \mid n_\nu=1, 2, \cdots ; \mathrm{m}_\nu=1, 2, \cdots, 2^{n_\nu-1}(\nu=1, 2, \cdots, k) ; k=1, 2, \cdots\}$ of disjoint Cantor sets.

$$\text{Set } F^{(1)}=E^{(0)},\ F^{(2)}=F^{(1)}\cup\left(\bigcup_{n_1=1}^{\infty}\bigcup_{m_1=1}^{2^{n_1-1}}E_{n_1,m_1}^{(1)}\right), \cdots$$

$$F^{(k)}=F^{(k-1)}\cup\left(\bigcup_{n_1=1}^{\infty}\bigcup_{m_1=1}^{2^{n_1-1}}\cdots\bigcup_{n_{k-1}=1}^{\infty}\bigcup_{m_{k-1}=1}^{2^{n_{k-1}-1}}E_{n_1,m_1;\cdots;n_{k-1},m_{k-1}}^{(k-1)}\right).$$

Finally set

$$S=S(p_n^{(\nu)} \mid n=1, 2, \cdots ; \nu=1, 2, \cdots)=I-\bigcup_{k=1}^{\infty}F^{(k)}.$$

Then S is a subset of I of the second category. Observe that it is the complement of $\bigcup_{k=1}^{\infty} F^{(k)}$ which is of the first category in I.

4. Let $U^{(k)}=I-F^{(k)}$. Then $\{U^{(k)}\}_{k=1}^{\infty}$ is a decreasing sequence of open sets and $S=\bigcap_{k=1}^{\infty} U^{(k)}$. Clearly

$$U^{(k)}=\bigcup_{n_1=1}^{\infty}\bigcup_{m_1=1}^{2^{n_1-1}}\cdots\bigcup_{n_k=1}^{\infty}\bigcup_{m_k=1}^{2^{n_k-1}}I_{n_1,m_1;\cdots;n_k,m_k}^{(k)}$$

and the length $l(I_{n_1,m_1;\cdots;n_k,m_k}^{k})$ of $I_{n_1,m_1;\cdots;n_k,m_k}^{(k)}$ is given by

$$2^{k-\sum_{\nu=1}^{k}n_k}\left(\prod_{\nu=1}^{k}p_1^{(\nu)}\cdots p_{n_\nu-1}^{(\nu)}\right)^{-1}\prod_{\nu=1}^{k}\left(1-\frac{1}{p_{n_\nu}^{(\nu)}}\right).$$

Since the logarithmic capacity $\gamma(I_{n_1,m_1;\cdots;n_k,m_k}^{(k)})$ of $I_{n_1,m_1;\cdots;n_k,m_k}^{(k)}$ is $l(I_{n_1,m_1;\cdots;n_k,m_k}^{(k)})/4$ (cf. Tsuji [3, p. 84]) and

$$1/\log\gamma(U^{(k)})^{-1}\leq\sum_{n_1=1}^{\infty}\sum_{m_1=1}^{2^{n_1-1}}\cdots\sum_{n_k=1}^{\infty}\sum_{m_k=1}^{2^{n_k-1}}1/\log\gamma(I_{n_1,m_1,\cdots,n_k,m_k}^{(k)})^{-1},$$

we infer $1/\log\gamma(U^{(k)})^{-1}\leq\sum_{n_1,\cdots,n_k=1}^{\infty}a_{n_1,\cdots,n_k}$, $a_{n_1,\cdots,n_k}$ being

$$\frac{2^{n_1+\cdots+n_k-k}}{(n_1+\cdots+n_k-k)\log 2+\sum_{\nu=1}^{k}(\log p_1^{(\nu)}+\cdots+\log p_{n_\nu-1}^{(\nu)})+\sum_{\nu=1}^{k}\log\frac{p_{n_\nu}^{(\nu)}}{p_{n_\nu}^{(\nu)}-1}+\log 4}.$$

Setting $p_i^{(\nu)}=e^{q_i^{(\nu)}}$, we obtain

$$1/\log \gamma(U^{(k)})^{-1} \le \sum_{n_1,\cdots,n_k=1}^{\infty} \frac{2^{n_1+\cdots+n_k-k}}{\sum_{\nu=1}^{k}(q_1^{(\nu)}+\cdots+q_{n_\nu-1}^{(\nu)})}$$
$$\le k!\sum_{n_1\le n_2\le\cdots\le n_k} \frac{2^{n_1+\cdots+n_k-k}}{\sum_{\nu=1}^{k}(q_1^{(\nu)}+\cdots+q_{n_\nu-1}^{(\nu)})}$$
$$\le k!\sum_{n_1\le n_2\le\cdots\le n_k} \frac{2^{kn_k-k}}{q_{n_k-1}^{(k)}}$$
$$\le k!\sum_{n=1}^{\infty} 2^{kn-k}n^k/q_{n-1}^{(k)}.$$

If we take $q_n^{(k)}=k!\,2^{2kn-k}n^k$, then

$$\gamma(S)\le\gamma(U^{(k)})\le e^{-2^k} \qquad (k=1,2,\cdots)$$

and hence $\gamma(S)=0$.

Therefore $S(p_n^{(k)}|n=1,2,\cdots;k=1,2,\cdots)$ with $p_n^{(k)}=e^{k!\,2^{2kn-k}n^k}$, for example, is a G_δ-subset of $[0,1]$ of the second category with zero logarithmic capacity.

5. Relating to the above example we append here two problems unknown to the authors:

i) We may call the set $\bigcup_{k=1}^{\infty}F^{(k)}$ the *iterated Cantor set* denoted by $C(p_n^{(\nu)}|n=1,2,\cdots,\nu=1,2,\cdots)$. The complete explicit condition on $\{p_n^{(\nu)}\}$ for its capacity to vanish is well-known. Seek the corresponding one for the complement $S(p_n^{(\nu)}|n=1,2,\cdots;\nu=1,2,\cdots)$ of $C(p_n^{(\nu)}|n=1,2,\cdots;\nu=1,2,\cdots)$ in I.

ii) Give nonconstructive proof as in 2 for the existence of a linear set of the second category with zero logarithmic capacity. Suitable substitution of Fatou theorem and functions of class (U) in 2 by Beurling-Tsuji theorem and functions with finite spherical areas might give the required.

References

[1] E. Collingwood: On sets of maximum indetermination of analytic functions. Math. Z., **67**, 377-396 (1957).
[2] K. Noshiro: Cluster Sets. Springer-Verlag (1960).
[3] M. Tsuji: Potential Theory in Modern Function Theory. Maruzen (1959).