138. A Note on Nets and Metrization

By Francis SIWIEC and Jun-iti NAGATA*)
University of Pittsburgh

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A collection $\mathcal{B}$ of subsets of a topological space X is a *net* for X if for each point x in X and open neighborhood U of x there exists a $B\in\mathcal{B}$ such that $x\in B\subset U$. A space with a σ-locally finite net is called a *σ-space* and a regular space with a countable net is called a *cosmic* space (A. Okuyama [15] and E. Michael [7]). We assume at least T_1 for every topological space throughout this paper. For terminology not defined here, see J. Nagata [11].

A collection $\mathcal{F}$ of closed subsets of a topological space X is a *ct-net* for X if for any different points x, x' of X there is an $F\in\mathcal{F}$ such that $x\in F$ and $x'\notin F$. A space with a σ-closure preserving *ct-net* is called a *$\sigma^{\#}$-space.*

A base $\mathcal{B}$ for a space X is *point countable* if each point x of X is in at most countably many members of $\mathcal{B}$.

A space is *semi-metrizable* if there is a distance function d for X such that (i) for each x and y in X, $d(x,y)=d(y,x)\geqslant 0$ and $d(y,y)=0$ only if $x=y$, (ii) for $x\in X$ and $M\subset X$, $\inf\{d(x,y)\mid y\in M\}=0$ iff $x\in\bar{M}$.

A space X has a *G_δ-diagonal* if the diagonal in $X\times X$ is a G_δ-set.

Let $\{\mathcal{U}_i\mid i=1,2,\cdots\}$ be a sequence of covers of a space X satisfying the condition:

(M) If $\{K_i\mid i=1,2,\cdots\}$ is a decreasing sequence of non-empty sets of X such that $K_i\in St(x_0,\mathcal{U}_i)$ for each i and for some fixed point x_0 of X, then $\bigcap_{i=1}^{\infty}\bar{K}_i\neq\phi$.

A space is a *$w\Delta$-space* if there exists a sequence $\{\mathcal{U}_i\}$ of open covers satisfying (M) (C. Borges [3]). A space is an *M-space* if there exists a normal sequence $\{\mathcal{U}_i\}$ of open covers satisfying (M). (K. Morita [8]). A space is an *M^*-space* if there exists a sequence $\{\mathcal{U}_i\}$ of locally finite closed covers satisfying (M) (T. Ishii [6]). A space is an *$M^{\#}$-space* if there exists a sequence $\{\mathcal{U}_i\}$ of closure-preserving closed covers satisfying (M).

The results of this paper have been partially announced in [17] and [18].

A. Okuyama [16] has shown that a collectionwise normal σ-space has a σ-discrete net. However we have the following

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1. Theorem. *For a regular space X the following are equivalent:*

i) *X has a σ-closure preserving net,*

ii) *X is a σ-space,*

iii) *X has a σ-discrete net.*

Proof. We show that a space X with a σ-closure preserving net has a σ-discrete net. Let $\mathcal{B}=\cup_{i=1}^{\infty}\mathcal{B}_i$ be a net for X where each $\mathcal{B}_i=\{B_\alpha|\alpha\in A_i\}$ is closure preserving. For arbitrary index $\alpha\in A_i$ and for arbitrary natural number k, let $F_{\alpha k}=\cup\{\bar{B}|B\in\mathcal{B}_k,\ \bar{B}\cap\bar{B}_\alpha=\phi\}$. For $A'\subset A_i$, let $G_k(A')=[\cap\{\bar{B}_\alpha|\alpha\in A'\}]\cap[\cap\{F_{\alpha k}|\alpha\in A_i-A'\}]$. Let $\mathcal{D}_{ik}=\{G_k(A')|A'\subset A_i\}$. Then for fixed i, k, $\mathcal{D}_{ik}$ is a disjoint collection. For suppose $G_k(A')\cap G_k(A'')\neq\phi$ for A', $A''\subset A_i$, $A'\neq A''$. Then there is an $\alpha\in A'-A''$, or vice versa. And there exists $x\in[\cap\{\bar{B}_\alpha|\alpha\in A'\}]\cap[\cap\{F_{\alpha k}|\alpha\in A_i-A'\}]$ and $x\in[\cap\{\bar{B}_\alpha|\alpha\in A''\}]\cap[\cap\{F_{\alpha k}|\alpha\in A_i-A''\}]$. So $x\in\bar{B}_\alpha\cap F_{\alpha k}$ where $\alpha\in A_i$, contradicting the choice of $F_{\alpha k}$. $\mathcal{D}_{ik}$ is easily seen to be closure preserving, therefore $\mathcal{D}_{ik}$ is discrete. And $\mathcal{D}=\cup_{i,k=1}^{\infty}\mathcal{D}_{ik}$ is a net. For if U is an open neighborhood of a point x in X, there is a $\mathcal{B}_i$ and $B_{\alpha_0}\in\mathcal{B}_i$ such that $x\in B_{\alpha_0}\subset\bar{B}_{\alpha_0}\subset U$. Since $\mathcal{B}_i$ is closure preserving, $U\{\bar{B}|B\in\mathcal{B}_i,\ x\notin\bar{B}\}$ is a closed set which does not contain x. So there is a $\mathcal{B}_k$ and $B_{\alpha_1}\in\mathcal{B}_k$ such that $x\in B_{\alpha_1}$ and $\bar{B}_{\alpha_1}\cap\cup\{\bar{B}|B\in\mathcal{B}_i,\ x\notin\bar{B}\}=\phi$. Let $A'=\{\alpha\in A_i|x\in\bar{B}_\alpha\}$. Then $x\in G_k(A')\subset U$.

Remark. It is easy to see, by use of this Theorem, that if f is a closed continuous mapping of a σ-space X onto a regular space Y, then Y is also a σ-space. This improves a result and simplifies the proof of A. Okuyama in [15].

2. Proposition. *A regular σ-space is a $\sigma^{\#}$-space and has a G_δ-diagonal.*

Proof. The G_δ-diagonal property was noted by Coban in [4].

3. Proposition. *A regular M-space with a G_δ-diagonal is first countable.*

Proof. Since the space X has a G_δ-diagonal, $\{(x,x)|x\in X\}=\cap_{i=1}^{\infty}G_i$ where each G_i is open in $X\times X$. By regularity for each x in X, there is a closed neighborhood $N_i'(x)$ of x such that $N_i'(x)\subset\cup\{V|x\in V,\ V\text{ is open in }X,\ V\times V\subset G_i\}$. Then for each x, $\cap_{i=1}^{\infty}N_i'(x)=\{x\}$. Let $\{\mathcal{U}_i|i=1,2,\cdots\}$ be a normal sequence of open covers satisfying (M). Let $N_i(x)=\operatorname{Int}N_i'(x)\cap St(x,\mathcal{U}_i)\cap N_{i-1}(x)$. To show that $\{N_i(x)|i=1,2,\cdots\}$ is a neighborhood base at x, let W be an open neighborhood of x. For each i, $\overline{N_{i+2}(x)}\subset\overline{St(x,\mathcal{U}_{i+2})}\subset\overline{U_{i+1}}$ for some $U_{i+1}\in\mathcal{U}_{i+1}$. And $\overline{U_{i+1}}\subset St(U_{i+1},\mathcal{U}_{i+1})\subset U_i$ for some $U_i\in\mathcal{U}_i$. So $\overline{N_{i+2}(x)}\subset\overline{U_{i+1}}\subset U_i\subset St(x,\mathcal{U}_i)$. Let $K_i=\overline{N_{i+2}(x)}-W$ and assume $K_i\neq\phi$ for every i. Then $\{K_i|i=1,2,\cdots\}$ is a decreasing sequence of sets such that $K_i \subset St(x, \mathcal{U}_i)$ for all i. Thus, by Condition (M), $\phi \neq \cap_{i=1}^{\infty} K_i = \cap_{i=1}^{\infty} K_i = \cap_{i=1}^{\infty} \overline{N_{i+2}(x)} - W = \{x\} - W = \phi$. Therefore for some i, $\phi = K_i = \overline{N_{i+2}(x)} - W$, and so $N_{i+2}(x) \subset W$.

4. **Proposition.** *A first countable space with a σ-closure preserving closed net is semi-metrizable.*

Proof. This follows from a theorem of Arhangelskii [1, Theorem 2.8] and Proof of Theorem 1.

5. **Proposition.** *A T_2, paracompact, $w\Delta$-space with a G_δ-diagonal is metrizable.*

Proof. Since a paracompact $w\Delta$-space is an M-space, this is a corollary of a theorem in [3], [13], or [17].

6. **Proposition.** *A T_2, semi-metrizable M-space is metrizable.*

Proof. This result is due independently to Nedev [12]. A T_2, semi-metrizable space has a G_δ-diagonal. For if we let $H_n(x) = \text{Int}\{y \mid d(x, y) < 1/n)$, and $G_n = \cup \{H_n(x) \times H_n(x) \mid x \in X\}$, then $\{(x, x) \mid x \in X\} = \cap_{n=1}^{\infty} G_n$. K. Morita showed [8] that for an M-space X there is a closed continuous mapping f of X onto a metrizable space Y such that $f^{-1}(y)$ is countably compact for each y in Y. If we can show that $f^{-1}(y)$ is compact for each y in Y, then since Y is paracompact, X is also paracompact and metrizable by Proposition 5. Since in a countably compact space every uncountable subset has a limit point, we show that a semi-metrizable space X' with this property is Lindelöf. Let $\mathcal{U} = \{U_\alpha \mid \alpha < \tau\}$ be a well ordered open cover of X'. For each $\alpha < \tau$ and natural number i, choose a point $p(\alpha, i)$, if possible, such that $S_{1/i}(p(\alpha, i)) \subset U_\alpha$ and if $p(\alpha, i) \in U_\beta \in \mathcal{U}$ then $\beta \geqslant \alpha$. Then $P_i = \{p(\alpha, i) \mid \alpha < \tau\}$ has no limit point. For if x is a limit point of P_i, choose the first $\gamma < \tau$ such that $x \in U_\gamma$. Let $F_\alpha = \{q \mid S_{1/i}(q) \subset U_\alpha,\ q \notin U_\beta$ for all $\beta < \alpha\}$ and $F = \cup \{F_\alpha \mid \alpha \neq \gamma\}$. Then $X' - \bar{F}$ is an open neighborhood of x which contains at most one point of P_i. Thus, since P_i has no limit point, P_i is countable. So $\{U_\alpha \mid p(\alpha, i)$ is defined for some $i\}$ is a countable subcover of X'.

7. **Lemma.** *A collectionwise normal, $M^\#$, $\sigma^\#$-space X is an M, σ-space.*

Proof. Let $\{\mathcal{F}_i \mid i = 1, 2 \cdots\}$ be a sequence of closure preserving closed covers of X satisfying (M) and $\cup_{i=1}^{\infty} \mathcal{D}_i$ a σ-closure preserving ct-net of X. We may assume each $\mathcal{D}_i$ is a cover of X. Put $\mathcal{H}_i = \mathcal{F}_i \wedge \mathcal{D}_i \wedge \mathcal{H}_{i-1}$, then $\{\mathcal{H}_i \mid i = 1, 2, \cdots\}$ is a sequence of closure preserving closed covers of X satisfying (M) and $\cup_{i=1}^{\infty} \mathcal{H}_i$ is a ct-net of X. Let U be an open neighborhood of a point x_0 of X. Then $C(x_0, \mathcal{H}_i) = \cap \{H \mid x_0 \in H \in \mathcal{H}_i\} \subset U$ for some i. Because otherwise $K_i = (X - U) \cap C(x_0, \mathcal{H}_i)$, $i = 1, 2, \cdots$, is a sequence of closed sets satisfying Condition (M) with respect to $\mathcal{H}_i$. Thus there is an $x \in \cap_{i=1}^{\infty} K_i$. This contradicts the fact that $\cup_{i=1}^{\infty}\mathcal{H}_i$ is a *ct*-net. Therefore $\cup_{i=1}^{\infty}\mathcal{H}'_i$, where $\mathcal{H}'_i=\{C(x, \mathcal{H}_i)\,|\,x \in X\}$, is a σ-closure preserving net of X and hence by Theorem 1, X is a σ-space. By A. Okuyama [14] a collectionwise normal σ-space is paracompact. Thus we can prove that X is an M-space by use of a method similar to that of K. Morita [9, Theorem 1.1].

8. Proposition. *An M, $\sigma^{\#}$-space is a σ-space.*

Proof. In Proof of Lemma 7 we did not use collectionwise normality to prove that X was σ.

9. Theorem. *The following conditions are equivalent:*

i) *X is metrizable,*

ii) *X is regular, a $\sigma^{\#}$-space, and an M-space,*

iii) *X is collectionwise normal, a $\sigma^{\#}$-space, and an $M^{\#}$-space,*

iv) *X is collectionwise normal, a σ-space, and a $w\Delta$-space,*

v) *X is collectionwise normal, a σ-space, and has a point countable base.*

Proof. Since i) obviously implies the other Conditions, we prove only that i) is derived from any of the other conditions. ii) follows from Propositions 8, 2, 3, 4, and 6. Lemma 7 and Condition ii) give Condition iii). For Condition iv) a collectionwise normal σ-space is paracompact. By Propositions 2 and 5, X is metrizable. Assuming Condition v), X is first countable since it has a point countable base. By Proposition 4, X is semi-metrizable. R. Heath [5] shows that a semi-metrizable space with a point countable base is developable. But a collectionwise normal developable space is metrizable by R. Bing [2].

Remark. This theorem is a generalization of J. Nagata [10, Theorem 3], and A. Okuyama [14, Theorem 3.6].

10. Theorem. *A space X is cosmic iff X is collectionwise normal, separable and a σ-space.*

Proof. We prove only the sufficiency. By use of Theorem 1, there is a σ-discrete net $\mathcal{A}$. Then $\mathcal{A}=\cup_{i=1}^{\infty}\mathcal{A}_i$, where each $\mathcal{A}_i=\{A_\lambda\,|\,\lambda \in \Lambda_i\}$ is discrete. Since X is collectionwise normal, for each i there is a disjoint collection $\mathcal{U}_i=\{U_\lambda\,|\,\lambda \in \Lambda_i\}$ of open sets such that $U_\lambda \supset A_\lambda$ for $\lambda \in \Lambda_i$. Since X is separable each $\mathcal{U}_i$ is countable and so also is $\mathcal{A}_i$. Therefore $\mathcal{A}$ is a countable net.

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