127. Π-imbeddings of Homotopy Spheres

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1. Let Θ_n be the group of homotopy n-spheres and let us consider the π-imbedding (i.e., imbedding with a trivial normal bundle) of $\tilde{S}^n \in \Theta_n$ in the $(n+k)$-dimensional euclidean space R^{n+k}.

If $\tilde{S}^n$ is π-imbeddable in R^{n+k}, then the connected sum $\tilde{S}^n \# \tilde{S}^n$ is also π-imbeddable in R^{n+k}. Thus we want to determine the smallest codimension with which the generator $\tilde{S}^n_0$ of Θ_n is π-imbeddable.

In [1], W. C. Hsiang, J. Levine, and R. H. Szczarba showed that every homotopy n-sphere is π-imbeddable in R^{n+k} if $k \geqq n-2$ for all n or $k > \dfrac{n+1}{2}$ for $n \leqq 15$. They also showed that $\tilde{S}^{16}_0 (\in \Theta_{16} \cong Z_2)$ is not π-imbeddable in R^{29}.

Now, owing to the classification theorem of S. P. Novikov [6], we can calculate the number of the differentiable structures of a direct product of spheres and this gives us some informations on our problem. In this way, we obtain the following results.

n	8	9	10	13	14	15	16	17
order of Θ_n	2	8	6	3	2	16256	2	16
order of $\Theta_n(\partial\pi)$	1	2	1	1	1	8128	1	2
k	4	4	4∼6	3∼4	?∼8	3∼4	14	?∼13

(k is the smallest codimension with which the generator of Θ_n is π-imbeddable.)

If $\Theta_n = 0$ or $\Theta_n(\partial\pi)$, then $k=1$ or 2 respectively ([3], Theorem I).

2. **Lemma 1.** *If $\tilde{S}^n (n \geqq 5)$ is π-imbeddable in R^{n+k}, then $\tilde{S}^n \times S^{k-1}$ is diffeomorphic to $S^n \times S^{k-1}$.*

The proof is the same as that of Theorem 5.2 in [7].

Conversely, we have

Lemma 2. *If $\tilde{S}^n \times S^{k-1}$ and $S^n \times S^{k-1}$ are diffeomorphic modulo a point, then $\tilde{S}^n$ is π-imbeddable in R^{n+k}.*

Proof. Since $(\tilde{S}^n \times S^{k-1}) \# \tilde{S}^{n+k-1}$ is diffeomorphic to $S^n \times S^{k-1}$ and $\tilde{S}^{n+k-1}$ can be summed to $\tilde{S}^n \times S^{k-1}$ away from $\tilde{S}^n \times x_0$ for some point $x_0 \in S^{k-1}$, the imbedding

$$\tilde{S}^n \subset (\tilde{S}^n \times S^{k-1}) \# \tilde{S}^{n+k-1} \approx S^n \times S^{k-1} \subset R^{n+k}$$

has a trivial normal bundle.

From these lemmas, we have

Corollary 3. *A necessary and sufficient condition for* $\tilde{S}^n \times S^{k-1}$ *and* $S^n \times S^{k-1}$ *to be diffeomorphic modulo a point is that they are diffeomorphic.*

Thus our problem is reduced to finding the smallest k such that $\tilde{S}_0^n \times S^{k-1}$ is diffeomorphic modulo a point to $S^n \times S^{k-1}$, and hence the classification theorem of Novikov is useful.

Let $E^{N-k+1};\ \pi_{n+k-1}(S^{k-1}) \longrightarrow G(n) = \pi_{n+N}(S^N)$ be the $(N-k+1)$-fold suspension homomorphism and let $J_n;\ \pi_n(SO_N) \longrightarrow G(n)$ be the stable J-homomorphism. Then the classification theorem applied to $S^n \times S^{k-1}$ ([6], Theorem 11.5) can be restated as follows.

Theorem 4. *If* $E^{N-k+1}(\pi_{n+k-1}(S^{k-1})) \cup J_n(\pi_n(SO_N)) = G(n)$, *then* $\tilde{S}_0^n$ *is π-imbeddable in* R^{n+k}.

3. As for the notations, we follow the usage in H. Toda [8].

The case $n=8$. Levine showed that $\tilde{S}_0^8$ is imbeddable in R^{12} and is not imbeddable in R^{11} ([4]).

Let λ be the generator of $\pi_7(SO_N) \cong Z$ and let η_7 be the generator of $\pi_8(S^7) \cong Z_2$, then $\pi_8(SO_N) = Z_2[\lambda \circ \eta_7]$. Since $J_7(\lambda)$ generates $G(7) = Z_{16}[\sigma] + Z_3[\alpha_2] + Z_5[\alpha_{1,5}]$, it follows that $J_8(\lambda \circ \eta_7) = \sigma \circ \eta = \varepsilon + \bar{\nu}$ ([8], Theorm 7.1).

On the other hand, $\pi_{11}(S^3) = Z_2[\varepsilon_3]$ where $E^{N-3}(\varepsilon_3) = \varepsilon$. Hence $E^{N-3}(\pi_{11}(S^3)) \cup J_8(\pi_8(SO_N)) = G(8) = Z_2[\varepsilon] + Z_2[\bar{\nu}]$ and it follows from Theorem 4 that $\tilde{S}_0^8$ is π-imbeddable in R^{12}.

The case $n=9$. Levine showed that $\tilde{S}_0^9$ is imbeddable in R^{13} and is not imbeddable in R^{12}.

Since $\pi_9(SO_N) = Z_2[\lambda \circ \eta_7 \circ \eta_8]$, it follows that $J_9(\lambda \circ \eta_7 \circ \eta_8) = \sigma \circ \eta^2 = \nu^3 + \eta \circ \varepsilon$, ([8], Lemmas 6.3 and 6.4).

On the other hand, $\pi_{12}(S^3) = Z_2[\mu_3] + Z_2[\eta_3 \circ \varepsilon_4]$ where $E^{N-3}(\mu_3) = \mu$ and $E^{N-3}(\eta_3 \circ \varepsilon_4) = \eta \circ \varepsilon$. Hence $E^{N-3}(\pi_{12}(S^3)) \cup J_9(\pi_9(SO_N)) = G(9) = Z_2[\nu^3] + Z_2[\mu] + Z_2[\eta \circ \varepsilon]$ and it follows that $\tilde{S}_0^9$ is π-imbeddable in R^{13}.

The case $n=10$. Levine showed that $\tilde{S}_0^{10}$ is imbeddable in R^{16} and is not imbeddable in R^{13}.

Although $\pi_{10}(SO_N) = 0$, we know that $E^{N-5};\ \pi_{15}(S^5) \longrightarrow G(10)$ is epimorphic ([8], Theorems 7.3 and 13.9). Hence $\tilde{S}_0^{10}$ is π-imbeddable in R^{16}. But $E^{N-4};\ \pi_{14}(S^4) \longrightarrow G(10)$ is not epimorphic, so we can not tell whether $\tilde{S}_0^{10}$ is π-imbeddable in R^{15} or not.

The case $n=13$. Since $E^{N-3}(\pi_{16}(S^3)) = G(13)$ ([8], Theorem 13.10), $\tilde{S}_0^{13}$ is π-imbeddable in R^{17}.

But we can not tell whether $\tilde{S}_0^{13}$ is π-imbeddable in R^{16} or not since $\pi_{13}(SO_N) = 0$ and $E^{N-2}(\pi_{15}(S^2)) = 0$.

The case $n=14$. It is known that $\tilde{S}_0^{14}$ is π-imbeddable in R^{22} ([1]). But we can not decrease the codimension by our method.

The case $n=15$. In this case, we know that $\pi_{15}(SO_N) \cong Z$ and the generator ρ of $G(15)=Z_{32}[\rho]+Z_2[\bar{\varepsilon}]+Z_3[\alpha_4]+Z_5[\alpha_{2,5}]$ belongs to Im J_{15} (see, for example, [5]).

On the other hand, we know that $E^{N-3}(\pi_{18}(S^3))=Z_2[\bar{\varepsilon}]+Z_3[\alpha_4]+Z_5[\alpha_{2,5}]$ ([8], Theorems 10.5, 13.10, and Proposition 13.6). Thus we have $E^{N-3}(\pi_{18}(S^3))\cup J_{15}(\pi_{15}(SO_N))=G(15)$, and it follows that $\tilde{S}_0^{15}$ is π-imbeddable in R^{19}.

But we can not tell whether $\tilde{S}_0^{15}$ is π-imbeddable in R^{18} or not.

The case $n=16$. We know that $\tilde{S}_0^{16}$ is π-imbeddable in R^{30} and there exists an imbedding in R^{29} with a non-trivial normal bundle ([1]). Since every imbedding is isotopic in this case, we have that $\tilde{S}_0^{16}$ is not π-imbeddable in R^{29}.

The case $n=17$. Since we know that E^{N-12}; $\pi_{29}(S^{12})\longrightarrow G(17)$ is epimorphic ([8], Theorem 12.17), $\tilde{S}_0^{17}$ is π-imbeddable in R^{30}. But we can not decrease the codimension by our method.

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