184. *Non-existence of Holomorphic Solutions of* $\partial u/\partial z_1=f$

By Isao WAKABAYASHI
Department of Mathematics, Tokyo Metropolitan University

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1. Consider the partial differential equation

$$\frac{\partial u}{\partial z_1}=f \tag{1}$$

on a domain D in the complex affine space $C^n(z_1, z_2, \cdots, z_n)$, where the given function $f=f(z_1, z_2, \cdots, z_n)$ is holomorphic in D. We are interested in global holomorphic solutions u of (1).

In particular, for $n=1$, it is well known that (1) has a global holomorphic solution for every f if and only if D is simply connected. We ask whether this is true for $n\geqslant 2$.

In what follows, we shall answer negatively this question. Namely, we shall give a domain D in C^3 which is holomorphically equivalent to a polycylinder (i.e., a product domain of disks) and on which (1) has no global solution for some holomorphic functions f.

For $n\geqslant 2$, a counterpart of simply connected domains is sometimes regarded as Runge domains.*) We shall give, however, a Runge domain $D\subset C^2$ on which (1) has no global solution for some holomorphic functions f.

On a convex domain in C^n, the existence theorem for global solutions of linear partial differential equations with constant coefficients was established by Harvey [2], and it was extended by Komatsu [3] to systems of those satisfying a compatibility condition. However convexity is a stronger condition than simply-connectedness. Moreover, as the case $n=1$ indicates, whether the simply-connectedness is sufficient or not for the existence of global solutions of such differential equations has been unknown for $n>1$.

2. Now we prove a proposition in order to show following Theorem 1.

Proposition. *Let D be a domain of holomorphy in $C^n(z_1, z_2, \cdots, z_n)$. If there exists a complex line L of the form $L=\{(z_1, z_2, \cdots, z_n)\in C^n \mid z_2=z_2^0, \cdots, z_n=z_n^0\}$ such that the intersection of L and D contains a multiply connected domain (in L), then (1) has no global solution on D for some holomorphic functions f.*

*) A domain of holomorphy in C^n is called a Runge domain if every holomorphic function in the domain can be uniformly approximated on an arbitrary compact set in the domain by polynomials.

Proof. Because $L\cap D$ contains a multiply connected domain, there exists a bounded set in the complement of $L\cap D$ with respect to L. Take an arbitrary point $(z_1^0, z_2^0, \cdots, z_n^0)$ belonging to such a set. Let f' be a function on $L\cap D$ defined by $f'(z_1, z_2^0, \cdots, z_n^0)=1/(z_1-z_1^0)$. Then f' is a holomorphic function on the analytic set $L\cap D$ in D. Hence, by Theorem B for domains of holomorphy, there exists a holomorphic function f on D whose restriction to $L\cap D$ is equal to f'. If there exists a global solution $u(z_1, \cdots, z_n)$ of (1) on D for f, we have

$$\frac{\partial u(z_1, z_2^0, \cdots, z_n^0)}{\partial z_1}=f(z_1, z_2^0, \cdots, z_n^0)=\frac{1}{z_1-z_1^0}.$$

Hence u must be multivalent. Consequently (1) has no global solution for the above function f. q.e.d.

Let F be a map of $C^3(x, y, z)$ into itself defined by $F(x, y, z)=(x, xy^2+z, xy-y+2yz)$, and let D_b denote a polycylinder

$$\{(x, y, z)\,|\,|x|<1+b, |y|<1+b, |z|<b, b>0\}.$$

Wermer showed [5]([1] p. 38) that for sufficiently small b, D_b and its image $F(D_b)$ are holomorphically equivalent by the map F, and $F(D_b)\cap\{(x, y, z)|y=1, z=0\}$ contains a circle $\{(x, y, z)\,|\,|x|=1, y=1, z=0\}$ without containing the point $(0, 1, 0)$. Hence, from the above proposition, we have

Theorem 1. *There exists a simply connected domain $D\subset C^3$ on which* (1) *has no global solution for some holomorphic functions f.*

3. We now consider Runge domains, and our result is the following:

Theorem 2. *There exists a Runge domain $D\subset C^2$ on which* (1) *has no global solution for some holomorphic functions f.*

Every componet of the intersection of an arbitrary complex line $L=\{(x, y)\in C^2(x, y)\,|\,ax+by+c=0\}$ and a Runge domain in C^2 is simply connected, where a, b, c are constant complex numbers. Hence the situation of this section differs from that of the preceding section.

Proof of Theorem 2. (*i*) *Construction of the domain.* In order to construct a domain with which we are concerned, let us consider the following function on $C^1(x)$ defined by $g(x)=x+c/x$, c being a constant complex number. By means of the function g, we shall form a closed bounded set $\sum$ in $C^2(x, y)$ in the following way:

$$\sum=\{(x, y)\in C^2\,|\,y=g(x),\ |g(x)|\leqslant 1,\ x\in C^1(x)\}.$$

By a fundamental theorem of Oka ([4] Théorème 1), for any neighborhood of $\sum$, there exists a Runge region (which may not be connected) included in the neighborhood and containing $\sum$. We may choose sufficiently small c so that the projection of $\sum$ to x-plane is a closed doubly connected domain not containing the origin, $\sum$ itself is connected, and the projection of $\sum$ to y-plane is a disk $\{y \in \boldsymbol{C}^1(y) \mid |y| \leqslant 1\}$. According to the above theorem of Oka, there exists a Runge domain D which does not contain $\{(x, y) \in \boldsymbol{C}^2 \mid x=0\}$. This Runge domain is what we wanted.

(*ii*) *A function f for which* (1) *has no solution.* Let f be a holomorphic function in the domain D defined by $f(x, y)=1/x$. Now, to show (1) has no global solution on D for f, assume the contrary, and denote a solution of (1) by $u(x, y)$. Let us consider a multivalent holomorphic function $u(x, y)-\log x$ on D. Then $u(x, y)-\log x$ is independent of the variable x, for

$$\frac{\partial\{u(x, y)-\log x\}}{\partial x}=0.$$

Hence we may denote the multivalent function by $h(y)$. The restriction of $h(y)$ to $\sum$ is regarded as a multivalent holomorphic function on the closed disk $\{y \in \boldsymbol{C}^1(y) \mid |y| \leqslant 1\}$. This is a contradiction. Therefore, on the domain D which is a Runge domain, and for the above function f, there exists no global solution of (1).

References

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