177. On Extension of Semifield Valued Linear Functionals

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(Comm. by Kinjirô KUNUGI, M. J. A., Oct. 12, 1968)

In their book [1], M. Antonovski, V. Boltjanski, and T. Sarymsakov introduced a new notion called topological semifield. The present author and S. Kasahara [2] obtained a theorem of Hahn-Banach type for a semifield valued functional. M. Kleiber and W. Pervin generalized our result in their paper [4].

In this note, we shall generalize a theorem by V. Klee [3].

Let E be a real linear space, and $\mathfrak{T}$ a set of linear transformations of E into E. Let p be a semifield valued subadditive functional on E, i.e., $p(x+y)\ll p(x)+p(y)$, $p(\alpha x)=\alpha p(x)$ for $\alpha\geq 0$.

For each $x\in E$, put

$$q(x)=\inf\{p(x+\sum_{i=1}^{k}T_iz_i)\,|\,T_i\in\mathfrak{T},\ y_i\in E,\ k \text{ is any positive integer}\}.$$

Let $f(x)$ be a semifield valued linear functional defined on a linear subspace E_f of E satisfying $f(x)\ll q(x)$. Then

$$0=f(0)\ll q(0)\ll p(\sum_{i=1}^{k}T_iz_i)\ll p(-x)+p(x+\sum_{i=1}^{k}T_iz_i).$$

Hence $0\ll p(-x)+q(x)$, which shows that q is well-defined on E. $q(x)$ is a semifield valued positive homogeneous functional. Let $x, y\in E$, then for any saturated neighborhood U of 0 in the semifield S, we can take V, T_i, z_i such that

$$p(x+\sum_{i=1}^{k}T_iz_i)\ll q(x)+V,$$
$$p(y+\sum_{i=k+1}^{n}T_iz_i)\ll q(y)+V,$$
$$V+V\subset U.$$

Therefore we have

$$\begin{aligned}q(x+y)&\ll p(x+\sum_{i=1}^{k}T_iz_i+y+\sum_{i=k+1}^{n}T_iz_i)\\&\ll p(x+\sum_{i=1}^{k}T_iz_i)+p(y+\sum_{i=k+1}^{n}T_iz_i)\\&\ll q(x)+q(y)+V+V\\&\ll q(x)+q(y)+U.\end{aligned}$$

Hence we have $q(x+y)\ll q(x)+q(y)$. By the Hahn-Banach type extension theorem for a semifield valued functional (see K. Iséki and S. Kasahara [1]), we can find a linear functional F satisfying $F=f$ on E_f, and $F\ll q$ on E. Then

$$F(Tz)\ll p(Tz+T(-z))=p(0)=0.$$

Hence we have $F(Tz)\ll 0$. Similarly we have $F(T(-z))\ll 0$. Therefore $FT=0$.

Then we have the following proposition which is a generalization of V. Klee result [3].

Proposition. *For a semifield valued subadditive, positive homogeneous functional p, put*

$$q(x)=\inf\{p(x+\sum_{i=1}^{k}T_iz_i)\mid T_i\in\mathfrak{T},\ z_i\in E,\ k \text{ is any positive integer}\}.$$

Then for a semifield valued linear functional f defined on a linear subspace E_f of E, $f\ll q$ on E_f if and only if there is a semifield valued linear functional F satisfying $F=f$ on E_f, $F\ll q$ on E and $FT=0$ on E for each $T\in\mathfrak{T}$.

Let E be a linear space with a semifield valued subadditive and positive homogeneous functional p. Let $\mathfrak{T}$ be a set of linear transformations of E into E, f a semifield valued linear functional on a linear subspace E_f of E satisfying $f\ll p$ on E_f and $T(D_f)\subset D_f$ and $f(T(x))=f(x)$ for every $T\in\mathfrak{T}$. Let F be a semifield valued linear functional which is an extension of f such that $F\ll p$ on E. From proposition, the existence of F is equivalent to: For $x\in E_f$, $T_i\in\mathfrak{T}$, and $z_i\in E$,

$$f(x)\ll p(x+\sum_{i=1}^{k}(T_i-I)z_i),$$

where I is the identity transformation on E.

Therefore we have the following result due to V. L. Klee [3].

Theorem 1. *A linear functional f satisfying $f\ll p$ mentioned above has a linear extension F such that $E_F=E$ if and only if for $x\in E_f$, $T_i\in\mathfrak{T}$, and $z_i\in E$,*

$$f(x)\ll p(x+\sum_{i=1}^{k}(T_i-I)z_i).$$

By the similar consideration in [3], we have Klee's theorem for semifield valued functions. Without any modification of the proofs of the real case, we can generalize many results on the extension of linear functionals in semifield valued functionals.

References

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