# 196. Note on Homogeneous Homomorphisms

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A homomorphism $\varphi$ of a semigroup $D$ onto a semigroup $D'$ is called homogeneous if each congruence class of $D$ induced by $\varphi$ has a same cardinal number. A homogeneous homomorphism will be called $h$-homomorphism.

Let $S$ be a set and $T$ be a semigroup. Consider a mapping $\Theta$ of $T\times T$ into the set of all binary operations defined on $S$, $(\alpha, \beta)\Theta=\theta_{\alpha,\beta}$, $(\alpha, \beta)\in T\times T$, such that

$$(x\theta_{\alpha,\beta}a)\theta_{\alpha\beta,\gamma}y=x\theta_{\alpha,\beta\gamma}(a\theta_{\beta,\gamma}y)$$

for all $\alpha, \beta, \gamma \in T$, all $a\in S$.

Let $S\times T=\{(x, \alpha)\,;\, x\in S, \alpha\in T\}$. Given $S$, $T$, $\Theta$, a binary operation is defined on $S\times T$ as follows:

(1) $$(x, \alpha)(y, \beta)=(x\theta_{\alpha,\beta}y, \alpha\beta).$$

Then $S\times T$ is a semigroup with respect to (1). The semigroup is called a general product of a set $S$ by a semigroup $T$ with respect to $\Theta$ and it is denoted by $S\overline{\times}_\Theta T$ or $S\overline{\times} T$. If a semigroup $D$ is isomorphic onto some $S\overline{\times}_\Theta T$, $|S|>1$, $|T|>1$, then $D$ is called general-product decomposable ($gp$-decomposable).

**Theorem 1.** *The following are equivalent:*

(2) *A semigroup D has a proper h-homomorphism.*

(3) *A semigroup D is gp-decomposable.*

(4) *There is a congruence $\rho$ on D and there is an equivalence $\sigma$ on D such that*

$$\rho\neq\omega,\quad \sigma\neq\omega,\quad \rho\cdot\sigma=\omega,\quad \rho\cap\sigma=\tau.$$

In Theorem 1, $\omega=D\times D$, $\tau=\{(x, x)\,;\, x\in D\}$ and $\rho\cdot\sigma=\{(x, y)\,;\, (x, z)\in\rho$ and $(z, y)\in\sigma$ for some $z\in D\}$.

The following theorem is concerned with the relationship between $h$-homomorphisms and homomorphisms.

**Theorem 2.** *If a semigroup D is homomorphic onto a semigroup T, there is a semigroup $\bar{D}$ such that*

(5) *D can be embedded into $\bar{D}$.*

(6) *$\bar{D}$ is h-homomorphic onto T and the homomorphism $\bar{D}\to T$ is the extension of the homomorphism $D\to T$.*

(7) *$\bar{D}\backslash D$ is an ideal of $\bar{D}$.*

*Also there is a semigroup $\bar{D}_1$ such that $\bar{D}_1$ satisfies* (5), (6), *and* (7′) *below:*

(7′) *$\bar{D}_1$ is an inflation of $D$.*

The concept of general product is the generalization of various concepts: direct product, Wreath product, semi-direct product, group extension and so on. For example free contents [3], [4], the semigroup of all binary operations on a set under a suitable operation [2], [5], and $\mathfrak{N}$-semigroups, i.e., commutative archimedean cancellative semigroups without idempotent are isomorphic to general products; commutative archimedean semigroups are the homomorphic images of general products [1].

The detailed proof of this paper will be published in [5] and all general products of a set $S$, $|S|=3$, by a right zero semigroup of order 2 will be published elsewhere; some special case of them will appear in [2].

## References

[1] T. Tamura: Construction of trees and commutative archimedean semigroups. Math. Nacht., **36**, 255–287 (1968).
[2] ——: Some contributions of computation to semigroups and groupoids. Proceedings of Conference on Computational Problems in Abstract Algebra, Pergamon Press (to be published).
[3] ——: The study of closets and free contents related to semilattice decomposition of semigroups. Proc. of Symposium of Semigroups, Academic Press (to be published).
[4] ——: On free content (to be published).
[5] ——: Basic study of general products (to be published).