229. *On Definitions of Boolean Rings and Distributive Lattices*

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G. R. Blakley, K. Iséki, and the present author give some new axioms for commutative rings and Boolean rings (see, [1]–[4]).

In this paper, we shall give new characterizations of Boolean rings and distributive lattices.

Theorem 1. *Let* $<X, 0, 1, +, ., ->$ *be an algebraic system containing* 0 *and* 1 *as elements of a set* X, *where* $+$ *and* $\cdot$ *are binary operations, and* $-$ *is a unary operation on* X (*we denote* $a\cdot b$ *by* ab). *Then* $<X, 0, 1, +, \cdot, ->$ *is a Boolean ring if it satisfies the following conditions*:

1) $r+0=r$,

2) $rl=r$,

3) $((-r)+r)a=0$,

4) $((ar+by)+cz)r=b(yr)+(ar+z(cr))$

for every a, b, c, r, y, z.

It is easily verified that every Boolean ring satisfies 1)–4).

Proof. The proof is divided into the following nine steps.

5) $(-r)+r$
$=((-r)+r)1$ {2}
$=0.$ {3}

6) $0a$
$=((-0)+0)a$ {5}
$=0.$ {3}

7) $a+b$
$=((a1+b1)+00)1$ {2, 1, 6}
$=b(11)+(a1+0(01))$ {4}
$=b+a.$ {2, 6, 1}

8) cz
$=((01+00)+cz)1$ {1, 7, 6, 2}
$=0(01)+(01+z(c1))$ {4}
$=zc.$ {6, 1, 7, 2}

9) $(b+a)+c$
$=(a+b)+c$ {7}
$=((a1+b1)+c1)1$ {2}
$=b(11)+(a1+1(c1))$ {4}

$=b+(a+c).$ {2, 8}

10) $(by)r$

$=((0r+by)+00)r$ {1, 6}

$=b(yr)+(0r+0(0r))$ {4}

$=b(yr).$ {6, 1}

11) $(b+c)r$

$=((0r+b1)+c1)r$ {2, 1, 6}

$=b(1r)+(0r+1(cr))$ {4}

$=br+cr.$ {2, 8, 6, 1}

12) r^2

$=((1r+00)+00)r$ {2, 8, 1}

$=0(0r)+(1r+0(0r))$ {4}

$=r.$ {6, 1, 2, 8}

12) For given a, b, the equation $a+x=b$ is solvable.

$a+((-a)+b)$

$=(a+(-a))+b$ {9}

$=b.$ {5, 1, 7}

Therefore $x=(-a)+b$.

Hence a set X is a ring and by 12), every element of X is the idempotent, therefore X is a Boolean ring.

Theorem 2. *Let* $<X, 0, 1, +, .>$ *be an algebraic system containing* 0 *and* 1 *as elements of a set* X, *where* $+$ *and* $\cdot$ *are binary operations on* X *(we denote* $a\cdot b$ *by* ab*). Then* $<X, 0, 1, +, \cdot>$ *is a distributive lattice, if it satisfies the following conditions*

1) $r+0=r,$

2) $r1=r,$

3) $0a=0,$

4) $((ar+by)+cz+d+d)r$
$=b(yr)+(ar+z(cr)+dr)$

for every a, b, c, d, r, y, z.

It is obvious that every distributive lattice satisfies 1)–4).

Proof. The proof is divided into the following eleven steps.

5) $a+b$

$=((a1+b1)+00+0+0)1$ {2, 1}

$=b(11)+(a1+0(01)+01)$ {4}

$=b+a.$ {2, 3, 1}

6) cz

$=((01+00)+cz+0+0)1$ {1, 5, 3, 2}

$=0(01)+(01+z(c1)+01)$ {4}

$=zc.$ {3, 1, 5, 2}

7) $(b+a)+c$

$=(a+b)+c$ {5}

$=((a1+b1)+(a1+1(c1)+01)$ $\{2,1\}$

$=b(11)+(a1+1(c1)+01)$ $\{4\}$

$=b+(a+c).$ $\{2,6,3,1\}$

8) $(by)r$

$=((0r+by)+00+0+0)r$ $\{3,1,5\}$

$=b(yr)+(0r+0(0r)+0r)$ $\{4\}$

$=b(yr).$ $\{3,1\}$

9) $(b+c)r$

$=((0r+b1)+c1+0+0)r$ $\{3,1,5,2\}$

$=b(1r)+(0r+1(cr)+0r)$ $\{4\}$

$=br+cr.$ $\{2,3,1,5\}$

10) $d+d$

$=((01+00)+00+d+d)1$ $\{3,1,5,2\}$

$=0(01)+(01+0(01)+d1)$ $\{4\}$

$=d.$ $\{3,1,5,2\}$

11) r^2

$=((1r+00)+00+0+0)r$ $\{2,6,1\}$

$=0(0r)+(1r+0(0r)+0r)$ $\{4\}$

$=r.$ $\{3,1,5,2,6\}$

Therefore a set X is a semiring and by 10), 11), we know that X is a distributive lattice.

References

[1] G. R. Blakley: Four axioms for commutative rings. Notices of Amer. Math. Soc., **15**, p. 730 (1968).

[2] K. Iséki: A simple characterization of Boolean rings. Proc. Japan Acad., **44**, 923–924 (1968).

[3] K. Iséki and S. Ôhashi: On definitions of commutative rings. Proc. Japan Acad., **44**, 920–922 (1968).

[4] S. Ôhashi: On axiom systems of commutative rings. Proc. Japan Acad., **44**, 915–919 (1968).