7. *Quotient and Bi-quotient Spaces of M-spaces*

By Jun-iti NAGATA[1)]
Department of Mathematics, University of Pittsburgh

(Comm. by Kinjirô KUNUGI, M. J. A., Jan. 13, 1969)

Since P. S. Alexandroff [1] posed a question 'which spaces can be represented as images of nice spaces under nice continuous mappings?', many interesting works have been done to answer the question. Among the works especially interesting are efforts to characterize the images of metric spaces by 'nice' continuous mappings. For examples, the image of a metric space by a continuous map (=mapping) f is characterized as a sequential space if f is a quotient map (S. Franklin [3]), a Fréchet space if f is a pseudo-open map (A. Arhangelskii [2]), and a first countable space if f is an open map[2)] (S. Hanai [4] and V. I. Ponomarev [10]). N. Lašnev [5] and E. Michael[3)] also characterized the image of a metric space by a closed (continuous) map and by a bi-quotient map, respectively. Thus it will be quite natural to try to characterize the continuous images of M-spaces which have recently emerged up as an important category of spaces including the countably compact spaces and the metric spaces. In the present paper we shall characterize the quotient image and bi-quotient image of an M-space and especially show that the quotient image of a paracompact M-space is nothing but an old k-space. It should be noted that the category of the quotient images of M-spaces contains all countably compact spaces as well as all sequential spaces.

Definition 1. A space X is called a *quasi-k-space* if a set F of X is closed iff (=if and only if) $F\cap C$ is closed in C for every countably compact set C of X.

Definition 2. A sequence $A_1\supset A_2\supset\cdots$ of subsets of a space X is called a *q-sequence* if any point sequence $\{x_i\mid i=1,2,\cdots\}$ satisfying $x_i\in A_i$ has a cluster point in $\bigcap_{i=1}^{\infty} A_i$. A sequence $U_1, U_2, \cdots$ of open nbds (=neighborhoods) of a point x of a space X is called a *q-sequence of nbds* if $U_1\supset \bar{U}_2\supset U_2\supset \bar{U}_3\supset\cdots$ and if any point sequence $\{x_i\mid i=1,2,\cdots\}$ satisfying $x_i\in U_i$ has a cluster point.

1) Supported by NSF Grant GP-5674.

2) As for terminologies and symbols in this paper, see J. Nagata [9]. Note that all spaces in this paper are at least Hausdorff, and all mappings are continuous.

3) The result is unpublished yet. The definition of bi-quotient map will be given later.

Lemma 1. *Let f be a continuous map from a space X onto a space Y. If $\{A_i \mid i=1, 2, \cdots\}$ is a q-sequence in X, then $\{f(A_i) \mid i=1, 2, \cdots\}$ is a q-sequence in Y.*

Theorem 1. *A regular space Y is a quasi-k-space iff there is a regular M-space[4] X and a quotient map f from X onto Y.*

Proof. Let f be a quotient map from a regular M-space X onto Y. Suppose V is a non-open set of Y. Then $f^{-1}(V)$ is non-open in X. Hence there is $x \in f^{-1}(V) - \operatorname{Int} f^{-1}(V)$. Since X is M, there is a q-sequence $U_1, U_2, \cdots$ of nbds of x. Note that $U_n \cap (x - f^{-1}(V)) \neq \phi$, $n=1, 2, \cdots$. Put $C(x) = \bigcap_{n=1}^{\infty} U_n = \bigcap_{n=1}^{\infty} \bar{U}_n$; then $C(x)$ is obviously a countably compact set.

i) Assume that $x \in \overline{C(x) \cap (x - f^{-1}(V))}$. (This necessarily implies that $C(x) \cap (X - f^{-1}(V)) \neq \phi$.) Then $f(C(x))$ is a countably compact set of Y such that $f(C(x)) \cap V$ is not open in $f(C(x))$. To see it let W be a given nbd of $f(x)$ in Y. Then $f^{-1}(W)$ is a nbd of x in X, and hence $f^{-1}(W) \cap C(x) \cap (X - f^{-1}(V)) \neq \phi$. This implies that $W \cap f(C(x)) \cap (Y - V) \neq \phi$ in Y. Therefore $f(x) \in f(C(x)) \cap V \cap \overline{f(C(x)) \cap (Y - V)}$ proving that $f(C(x)) \cap V$ is non-open in $f(C(x))$.

ii) Let us assume this time that $x \notin \overline{C(x) \cap (X - f^{-1}(V))}$. (Thus $C(x) \cap (x - f^{-1}(V))$ might be empty.) Then, since X is regular, there is an open nbd U of x such that

(1) $$\bar{U} \cap C(x) \cap (X - f^{-1}(V)) = \phi.$$

Observe that $U_n \cap U$ is a nbd of x. Since $x \in \overline{X - f^{-1}(V)}$, $U_n \cap U \cap (X - f^{-1}(V)) \neq \phi$, $n=1, 2, \cdots$ follows. Therefore we can choose

(2) $$x_n \in U_n \cap U \cap (X - f^{-1}(V)).$$

Since $\{U_n \mid n=1, 2, \cdots\}$ is a q-sequence $\{x_n\}$ has a cluster point x_0. It is obvious that $x_0 \in C(x) \cap \bar{U}$. This combined with (1) implies that $x_0 \in f^{-1}(V)$. Now, let $\overline{\{x_n\}} = K$; then we obtain

(3) $$x_0 \in K \cap f^{-1}(V).$$

To show that K is a countably compact set, suppose $\mathcal{U}$ is a countable open cover (=covering) of K. Then $K \cap C(x)$, as a countably compact set, is covered by finitely many members of $\mathcal{U}$, say $U_1, \cdots, U_k$. Observe that these $U_1, \cdots, U_k$ cover the whole K but finitely many of x_n, $n=1, 2, \cdots$. Because, if $x_{i_1}, x_{i_2}, \cdots \notin U_1 \cup \cdots \cup U_k$, then $\{x_{i_l} \mid l=1, 2, \cdots\}$ has a cluster point x'. It is obvious that $x' \in C(x) \cap K$. But this means that $U_1 \cup \cdots \cup U_k$ is a nbd of x', which is a contradiction. Thus K is covered by finitely many members of $\mathcal{U}$ proving that K is countably compact. Thus $f(k)$ is a countably compact set of Y, and

4) K. Morita's [8] terminology. He proved that a space is an M-space (paracompact M-space) iff it is the inverse image of a metric space by a quasi-perfect map (perfect map), where a closed continuous map is called quasi-perfect if for any point y of the range space $f^{-1}(y)$ is countably compact.

satisfies $f(x_0) \in f(K) \cap V$ because of (3). Suppose W is a given nbd of $f(x_0)$; then $f^{-1}(W)$ is a nbd of x_0 in X. Therefore by (3) $x_n \in f^{-1}(W)$ for some n, and hence $f^{-1}(W) \cap K \cap (X - f^{-1}(V)) \neq \phi$ follows from (2). This implies that $W \cap f(K) \cap (Y-V) \neq \phi$, i.e. $f(x_0) \in \overline{f(K) \cap (Y-V)}$. Thus $f(K) \cap V$ is non-open in $f(K)$. After all we have proved that Y is a quasi-k-space.

Conversely, suppose Y is a given regular quasi-k-space. Then we denote by $\{C_\alpha \mid \alpha \in A\}$ the collection of all countably compact sets of Y. Let X be the discrete sum of C_α, $\alpha \in A$. Then it is obviously a regular M-space. By combining the injections $C_\alpha \to Y$ we get a quotient map from X onto Y.

In view of the above proof we realize that we have practically proved the following.

Corollary. *For a regular space X the following statements are equivalent*:

i) *X is a quasi-k-space,*

ii) *X is a quotient space of a regular q-space,*[5]

iii) *X is a quotient space of a regular M-space,*

iv) *X is a quotient space of a regular, locally countably compact space.*

Theorem 2. *A space Y is a k-space iff there is a paracompact M-space X and a quotient map from X onto Y.*

Proof. All we have to do is to review the proof of Theorem 1 recalling that every paracompact, countably compact set is compact.

Corollary. *For a space X the following statements are equivalent*:

i) *X is a k-space,*

ii) *X is a quotient space of a paracompact q-space,*

iii) *X is a quotient space of a paracompact M-space,*

iv) *X is a quotient space of a locally compact space.*

The equivalence of i) and iv) was proved by D. E. Cohen, so we may say the corollary is an extension of Cohen's theorem.

Now, let us turn to images of M-spaces by bi-quotient maps. According to E. Michael [7], a continuous map f from X onto Y is called *bi-quotient* if for any $y \in Y$ and any cover $\mathcal{U}$ of $f^{-1}(y)$ by open sets of X, a finite subcollection of $f(\mathcal{U})$ covers a nbd of y in Y. A good property of bi-quotient maps is that any product of bi-quotient maps is a bi-quotient map. Every bi-quotient map is a quotient map. Also note that, as proved by Michael, a continuous map f from X onto Y is bi-quotient iff for any filter basis $\mathcal{G}$ in Y and any cluster point y of $\mathcal{G}$ there is $x \in f^{-1}(y)$ such that x is a cluster point of

5) See E. Michael [6].

$f^{-1}(\mathcal{G})=\{f^{-1}(G)\mid G\in\mathcal{G}\}$.

Definition 3. A space X is called a *bi-k-space* if any maximal filter $\mathcal{F}$ which converges to x in X contains a sequence $F_1, F_2, \cdots$ of members which form a q-sequence satisfying $x\in\bigcap_{i=1}^{\infty}F_i$.

Theorem 3. *A space Y is a bi-k-space iff there is an M-space X and a bi-quotient map f from X onto Y.*

Proof. Let f be a bi-quotient map from an M-space X onto Y and $\mathcal{G}$ a maximal filter converging to y in Y. Then there is $x\in f^{-1}(y)$ which is a cluster point of $f^{-1}(\mathcal{G})$. Since X is M, there is a q-sequence $U_1, U_2, \cdots$ of nbds of x. Note that $U_i\cap f^{-1}(G)\neq\phi$ for all i and for all $G\in\mathcal{G}$. Therefore $f(U_i)\cap G\neq\phi$ for all $G\in\mathcal{G}$ proving that $f(U_i)\in\mathcal{G}$. Thus by Lemma 1 we can conclude that Y is bi-k.

Conversely, let Y be a given bi-k-space. Suppose $\alpha=\{A_1, A_2, \cdots\}$ is an arbitrary q-sequence in Y with $\bigcap_{i=1}^{\infty}A_i\neq\phi$. Then we put $C(\alpha)=\bigcap_{i=1}^{\infty}A_i$ and define a topological space Y_α by introducing a stronger topology into Y. Namely we define a new nbd basis $\mathcal{N}(x)$ for each point x of Y by $\mathcal{N}(x)=\{A_i\cap U(x)\mid i=1, 2, \cdots;\ U(x)$ is a nbd of x in $Y\}$ for $x\in C(\alpha)$, $\mathcal{N}(x)=\{\{x\}\}$ for $x\in Y_\alpha-C(\alpha)$, and denote by Y_α thus defined space. For each natural number i we define an open cover $\mathcal{U}_{i,\alpha}$ of Y_α by $U_{i,\alpha}=\{A_i, \{z\}\mid z\in Y_\alpha-A_i\}$. Now, let X be the discrete sum of Y_α, $\alpha\in\Omega$, where Ω denotes the collection of all q-sequences α in Y with non-empty intersection. Define an open cover $\mathcal{U}_i$ of X by $\mathcal{U}_i=\cup\{\mathcal{U}_{i,\alpha}\mid\alpha\in\Omega\}$. Suppose $\{x_i\}$ is a sequence of points of X such that $x_i\in S(x_0, \mathcal{U}_i)$, $i=1, 2, \cdots$. Then $x_0\in Y_\alpha$ for some α. If $x_0\notin C(\alpha)$, then x_0 is a cluster point of $\{x_i\}$. If $x_0\in C(\alpha)$, then $x_i\in A_i$, $i=1, 2, \cdots$. This implies that $\{x_i\}$ regarded as a point sequence of Y has a cluster point $x'\in C(\alpha)$. Therefore it is easily seen that x' is a cluster point of $\{x_i\}$ in X, too. Thus X is an M-space. We define a map f from X onto Y by combining the identity maps from Y_α onto Y. Then f is continuous, because the topology of Y_α is stronger than that of Y. The only problem is to prove that f is bi-quotient. Let $\mathcal{G}$ be a filter basis in Y and y a cluster point of $\mathcal{G}$. Then there is a maximal filter $\mathcal{G}'$ which contains $\mathcal{G}$ and converges to y. Since Y is a bi-k-space, there is a q-sequence $A_1\supset A_2\supset\cdots$ such that $A_i\in\mathcal{G}'$, $i=1, 2, \cdots$, $y\in\bigcap_{i=1}^{\infty}A_i$. Let $\alpha=\{A_1, A_2, \cdots\}$; then $f^{-1}(y)\cap Y_\alpha$ consists of a single point which we denote by x. Let $A_i\cap U(x)$ be a basic nbd of x in Y_α. Regarding A_i and $U(x)$ subsets of Y, we observe that $A_i\in\mathcal{G}'$ and $U(x)\in\mathcal{G}'$ which imply $A_i\cap U(x)\in\mathcal{G}'$. Therefore $A_i\cap U(x)\cap G\neq\phi$ for every $G\in\mathcal{G}$. This proves that $A_i\cap U(x)\cap f^{-1}(G)\neq\phi$ in X. In other words x is a cluster point of $f^{-1}(\mathcal{G})$ in X. Hence f is a bi-quotient map.

Problem. Characterize images of M-spaces by pseudo-open maps, open maps and closed maps.

References

[1] P. S. Alexandroff: On some results concerning topological spaces and their continuous mappings. Proc. Prague Symposium, 41–54 (1962).

[2] A. B. Arhangelskii: Some types of factor mappings and the relations between classes of topological spaces. Doklady Akad. Nauk SSSR, **153** (1963); Soviet Math., **4**, 1335–1338 (1963).

[3] S. Franklin: Spaces in which sequences suffice. Fund. Math., **57**, 107–115 (1965).

[4] S. Hanai: On open mappings. II. Proc. Japan Acad., **37**, 233–238 (1961).

[5] N. Lašnev: Closed images of metric spaces. Dokl. Akad. Nauk SSSR, **170**, (1966); Soviet Math., **7**, 1219–1221 (1966).

[6] E. Michael: A note on closed maps and compactness. Israel J. Math., **2**, 173–176 (1964).

[7] ——: Bi-quotient maps and cartesian products of quotient maps (to appear).

[8] K. Morita: Products of normal spaces with metric spaces. Math. Ann., **154**, 365–382 (1964).

[9] J. Nagata: Modern General Topology. Amsterdam-Groningen (1968).

[10] V. I. Ponomarev: Axioms of countability and continuous mappings. Bull. Acad. Polon. Math. Ser., **8**, 127–134 (1960).