5. On Generalized Commuting Properties of Metric Automorphisms. II

By Nobuo AOKI
Department of Mathematics, Josai University, Saitama

(Comm. by Kinjirô KUNUGI, M. J. A., Jan. 13, 1969)

We study properties of ergodicity, totally ergodicity and mixing for the second class $C_2(T)$ of the generalized T-commuting order when T is ergodic metric automorphism with discrete spectrum. We use notations of [2]. In this paper results were first obtained by Hahn [3]. A metric automorphism S is said to have *continuous spectrum* if the only proper value of V_S is the number one and it is simple, and to have *infinite Lebesgue spectrum* if $L^2(X)$ has an orthonormal base $\{f_{i,n}: n=0,1,2,\cdots; i\in$ [infinite index set]$\}$ where $V_S f_{i,n}=f_{i,n+1}$ a.e. A countable sequence $E_1, E_2, \cdots$ of X is called a *separating sequence* if for every pair of x, y in X with $x\neq y$ there exists an integer n satisfying $x\in E_n$, $y\in X\backslash E_n$. If two automorphisms on a finite measure space (X, Σ, m) which contains a separating sequence $E_1, E_2, \cdots$ of measurable sets induce the same metric automorphism, then they differ on at most a set of measure zero [5]. Let G' be the group of all automorphisms of X with the identity I. We define as in [1], $C_0'(T)=\{S\in G': S=I \text{ a.e.}\}$ and n-th class $C_n'(T)=\{S\in G': T^{-1}S^{-1}TS\in C_{n-1}'(T)\}$, $n=1, 2, \cdots$ of the generalized T-commuting order for an ergodic automorphism T which has discrete spectrum.

Proposition 1. *Let (X, Σ, m) be a finite measure space which contains a separating sequence of measurable sets. If an automorphism T is totally ergodic and has discrete spectrum, then $C_1'(T)\neq C_2'(T)=C_3'(T)$. Furthermore, $C_0'(T)$, $C_1'(T)$, and $C_1'(T)$ are subgroups of G'.*

Proof. We denote by $\tilde{S}:\tilde{E}\to S^{-1}E$ ($\tilde{E}$ an element of the measure algebra and E a copy of $\tilde{E}$) the metric automorphism on the measure algebra induced by $S\in G'$. Let $\tilde{G}$ be a set $\{\tilde{S}: S\in G'\}$ and let $C_0(\tilde{T})[C_n(\tilde{T}), n=1, 2, \cdots]$ be a set $\{I\}$ a set $[\{\tilde{S}\in\tilde{G}: \tilde{S}\tilde{T}\tilde{S}^{-1}T^{-1}\in C_{n-1}(\tilde{T})\}, n=1, 2, \cdots]$. Then by [2] we see that $C_2(\tilde{T})=C_3(\tilde{T})$, and that $C_0(\tilde{T})$, $C_1(\tilde{T})$, and $C_2(\tilde{T})$ are subgroups of $\tilde{G}$. Since (X, Σ, m) contains a separating sequence of measurable sets, we can conlcude that $C_2'(T)=C_3'(T)$, and that $C_0'(T)$, $C_1'(T)$, and $C_2'(T)$ are subgroup of G'.

Let T be an ergodic metric automorphism with discrete spectrum. Then for every $S_2\in C_2(T)$ there exist metric automorphisms W, S such that W has each function of $\boldsymbol{O}(T)$ as proper function and the linear isometry V_S induced by S maps $\boldsymbol{O}(T)$ onto itself, and $S_2=SW(*)$[2].

Proposition 2. *Let T be an ergodic metric automorphism with discrete spectrum. If $S_2(\in C_2(T))$ is totally ergodic and if $S_2TS_2^{-1}T^{-1}$ is ergodic, then S_2 has infinite Lebesgue spectrum.*

Proof. For $S_2 \in C_2(T)$, we have $S_2=SW$ for S, W satisfying conditions of $(*)$. Suppose $f \in \boldsymbol{O}(T)$ and $V_S f=f$ a.e. Then we have $V_{S_2}f=\alpha_w(f)f$ a.e. and $V_{S_2TS_2^{-1}T^{-1}}f=f$ a.e. Thus $f=$constant a.e. since $S_2TS_2^{-1}T^{-1}$ is ergodic. Using condition of totally ergodicity of S_2 we see that S is ergodic, and that every function in $\boldsymbol{O}(T)$ contains only infinite orbits under V_S. Thus S_2 has infinite Lebesgue spectrum by ([4], p. 53).

Let V_S be an automorphism of $\boldsymbol{O}(T)$ onto itself. Suppose that $V_S^n f=f$ a.e. implies $n=1$ for $f \in \boldsymbol{O}(T)$. If $\boldsymbol{O}(T)$ contains an infinite orbit under V_S, then $\boldsymbol{O}(T)$ contains infinitely many such orbits [6].

Proposition 3. *Let T be an ergodic metric automorphism with discrete spectrum. Then $S_2(\in C_2(T))$ has continuous spectrum in the orthogonal complement $H^\perp$ of the subspace H in which S_2 has discrete spectrum. If $S_2(\in C_2(T)\setminus C_1(T))$ is totally ergodic, then S_2 has infinite Lebesgue spectrum in $H^\perp$.*

Proof. If $S_2 \in C_1(T)$, then S_2 has discrete spectrum. Suppose that $S_2 \notin C_1(T)$ and $S_2 \in C_2(T)$, then we have $S_2=SW$ for $S \neq I$, W satisfying conditions of $(*)$. Let H be the subspace spanned by the set of all $f \in \boldsymbol{O}(T)$ which have finite orbits under V_S. The space H is decomposed into the directed sum of $H(f)$ spanned by the orbit $f, V_S f, \cdots, V_S^{n-1}f$. We have $V_{S_2}(V_S^i f)=\alpha_W(V_S^i f)V_S^{i+1}f$ a.e., $i=0,1,\cdots,n-1$. Let $[V_{S_2}]$ be the matrix determined by the restriction of V_{S_2} to $H(f)$. Then $\det([V_{S_2}]-\lambda E)$ is given by $(-1)^n\lambda^n+\det([V_{S_2}])$. If $\lambda_0, \lambda_1, \cdots, \lambda_{n-1}$ are the proper values, then we have $\det([V_{S_2}]-\lambda_j E)=0$ for $j=0, 1, \cdots, n-1$ where E is the unit matrix. Thus there exists a non-zero function g_j such that $V_{S_2}g_j=\lambda_j g_j$ a.e. Since proper functions with different proper values are orthogonal in $H(f)$, $\{g_j: j=0,1,\cdots,n-1\}$ is a base of $H(f)$. We have shown that S_2 has discrete spectrum in $H(f)$. Let $O(f_\alpha)$ be an infinite orbit of f_α under V_S. If H' is spanned by $\bigcup_\alpha O(f_\alpha)$ then S_2 has continuous spectrum in H'. It turns out that $L^2(X)=H\oplus H'$. Therefore $H'=H^\perp$. If $S_2(\in C_2(T)\setminus C_1(T))$ is totally ergodic, then $\boldsymbol{O}(T)$ contains an infinite orbit under V_S. Thus $\boldsymbol{O}(T)$ contains infinitely many distinct such orbits $\{O(f_\alpha)\}$. Therefore S_2 has infinite Lebesgue spectrum in $H^\perp=\overline{\text{span} \bigcup_\alpha O(f_\alpha)}$.

Let T be an ergodic metric automorphism with discrete spectrum. For every $S_2 \in C_2(T)$, we have $S_2=SW$ for S, W satisfying conditions of $(*)$. Let $F(S)$ be a set $\{f \in O(T): f \text{ periodic under } V_S\}$ and let $\Lambda(W)$ be a set $\{\alpha_W(f): f \in O(T)\}$. Then S is ergodic if and only if $F(S)=\{1\}$, $\Lambda(W)$ is a subgroup of a circle group.

Proposition 4. *Let $S_2=SW$ belonging to $C_2(T)$ (S and W satisfy-*

ing conditions of $(*)$) *has not continuous spectrum. Then* S_2 *is ergodic if and only if we have the proper value* $\alpha_{S_2^n}(f)\neq 1$ *of* S_2^n *for each* $f \in F(S)$ *which is period* $n \neq 1$.

Proof. If $f \in F(S)$ with $f \neq 1$ a.e., then there exists an integer n such that $V_S^n f = f$ a.e. Therefore $V_{S_2}^n f = \alpha_{S_2^n}(f) f$ a.e. where $\alpha_{S_2^n}(f) = \alpha_w(f)\alpha_w(V_S f)\cdots\alpha_w(V_S^{n-1} f)$. Suppose $\alpha_{S_2^n}(f)=1$. Then we have $V_{S_2}^n f = f$ a.e. and $h=\sum_{k=0}^{n-1} V_{S_2}^k f \neq$ constant a.e. Therefore S_2 is not ergodic since $V_{S_2} h = h$ a.e. Conversely, suppose that $\alpha_{S_2^n}(f)\neq 1$ for each $f \in F(S)$ which is period $n \neq 1$ and let $V_{S_2} h = h$ a.e. for $h \in L^2(X)$. Consider the Fourier expansion $h = \sum_i \langle h, f_i\rangle f_i$ a.e. ($f_i \in \boldsymbol{O}(T)$). Then for $f_i \notin F(S)$ with $f_i \neq 1$ a.e., comparing coefficients of expansions of h and $V_{S_2} h (=\sum_i \langle h, f_i\rangle\ V_{S_2} f_i$ a.e.), we have $\langle h, V_S f_i\rangle = \alpha_w(f_i)\langle h, f_i\rangle$. Thus we obtain $\langle h, f_i\rangle = 0$ since the coefficients are square summable. For $f_k \in F(S)$ which is period $n \neq 1$, let $\alpha_{S_2^n}(f_k)$ be the proper value of S_2^n for f_k, then we have $\alpha_{S_2^n}(f_k)\langle h, f_k\rangle = \langle h, V_{S_2^n} f_k\rangle = \langle h, f_k\rangle$. But we obtain $\langle h, f_k\rangle = 0$ since $\alpha_{S_2^n}(f_k)\neq 1$. Therefore S_2 is ergodic.

Proposition 5. *Let* T *be an ergodic metric automorphism with discrete spectrum. Suppose that* $S_2 = SW$ *belonging to* $C_2(T)$(S *and* W *satisfying conditions of* $(*)$) *is ergodic, and that* $\Lambda(W)$ *contains no element of finite order except the unit element. Then* S_2 *is totally ergodic if and only if for* $f \in \boldsymbol{O}(T)$, f *is a proper function of* S_2^n *for some integer* $n \neq 0$ *then* f *is a proper function of* S_2.

Proof. Suppose now that S_2 is totally ergodic. Then it follows that for $f \in \boldsymbol{O}(T)$ $V_S^n f = f$ a.e. implies $V_S f = f$ a.e. Therefore f being a proper function of S_2^n is a proper function of S_2. Conversely, it is clear from ergodicity of S_2 that S_2 is totally ergodic.

Remark. Let T be an ergodic metric automorphism with discrete spectrum and let W be a metric automorphism which has every $f \in \boldsymbol{O}(T)$ as its proper function and let S be a metric automorphism which V_S maps $\boldsymbol{O}(T)$ onto itself. Then S has infinite Lebesgue spectrum if and only if SW has infinite Lebesgue spectrum.

References

[1] L. M. Abramov: Metric automorphisms with quasi-discrete spectrum. Amer. Math. Soc. Transl., **39** (2), 37–56 (1964).
[2] N. Aoki: On generalized commuting properties of metric automorphisms. I. Proc. Japan Acad., **44** (6), 467–471 (1968).
[3] F. J. Hahn: On affine transformations of compact abelian groups. Amer. J. of Math., **85** (3), 428–446 (1963).
[4] P. R. Halmos: Lectures on Ergodic Theory. Math. Soc. Japan (1956).
[5] P. R. Halmos and J. von Neumann: Operator methods in classical mechanics. II. Ann. of Math., **43** (2), 332–350 (1942).
[6] A. H. M. Hoare and W. Parry: Semi-groups of affine transformations. Oxford Quart. J. Math., **17** 106–111 (1966).