4. *On the Classical Flows with Discrete Spectra*

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Introduction. The main purpose of this paper is to consider the problem of classification or isomorphism of classical flows. Namely, in a certain class of classical flows, it is shown that their spectra and the order of differentiability of their eigenfunctions are the complete invariants (Theorem 2, § 3). This is an analogue of the famous theorem due to von Neumann: unitary equivalence of abstract flows implies their metrical equivalence in the case of ergodic abstract flows with discrete spectra.

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§ 1. Preliminaries. In this section we summarize necessary definitions and theorems. For details, refer to [1], [2].

Definition 1. *Classical flow* means the triple (M, μ, φ_t) (or briefly (φ_t)) formed by a C^∞-manifold M, a finite measure on M defined by a positive continuous density (we assume that $\mu(M)=1$) and one-parameter group (φ_t) of diffeomorphisms of M which preserve the measure μ.

Definition 2. Let (M, μ, φ_t) and (N, ν, ψ_t) be classical flows. (M, μ, φ_t) is *C^p-isomorphic* to (N, ν, ψ_t) as classical flows, when there exists a C^p-diffeomorphism $\iota: M \to N$ such that $\iota\circ\varphi_t=\psi_t\circ\iota$ for all t, and $\iota(\mu)=\nu$. We denote it by:

$$(M, \mu, \varphi_t) \underset{C^p}{\approx} (N, \nu, \psi_t).$$

Definition 3. A flow (φ_t) is called *ergodic*, when the condition

$$\mu\{\varphi_t A \ominus A\}=0 \text{ for all } t \text{ implies } \mu(A)=0$$

or $\mu(A)=1$, where $A\ominus B$ denotes the symmetric difference of two sets A and $B: A\ominus B=A\cup B-A\cap B$.

Let (M, μ, φ_t) be a classical flow, then it induces naturally a one-parameter group of unitary operators $\{U_t\}$ on the Hilbert space $\boldsymbol{H}=L^2(M, \mu)$ of complex valued square summable functions defined on M:

$$(U_t f)(x)=f(\varphi_t x), \text{ for } f\in \boldsymbol{H}.$$

By the decomposition theorem of Stone, these U_t have the following spectral resolution:

$$U_t=\int_{-\infty}^{+\infty} e^{2\pi i\lambda t} dE(\lambda),$$

where $\{E(\lambda)\}$ is a resolution of identity of $\boldsymbol{H}$.

Definition 4. Let $\boldsymbol{H}^{(\lambda)}=\{E(\lambda)-E(\lambda-0)\}\boldsymbol{H}$. We call λ an eigenvalue of the flow (φ_t), when $\dim \boldsymbol{H}^{(\lambda)}>0$ and an element of $\boldsymbol{H}^{(\lambda)}$ an eigenfunction of the flow (φ_t).

Let (M, μ, φ_t) and (N, ν, ψ_t) be classical flows, and, $\{U_t\}$ and $\{V_t\}$ be one-parameter groups of unitary operators which are induced from (M, μ, φ) and (N, ν, ψ_t) respectively. Then it is obvious that if (M, μ, φ_t) and (N, ν, ψ_t) are C^ρ-isomorphic as classical flows, then $\{U_t\}$ and $\{V_t\}$ are unitarily equivalent, especially their eigenvalues coincide.

Now the following results are well known in the ergodic theory.

Proposition 1. *A flow (φ_t) is ergodic if and only if $\lambda=0$ is a simple eigenvalue.*

Proposition 2. *Let (M, μ, φ_t) be classical flow and Λ^ρ be the set of eigenvalues of (φ_t) whose eigenfunctions are C^ρ-differentiable, then Λ^ρ is an additive subgroup of the real number group $\boldsymbol{R}$.*

§2. Quasi-periodic motions. Now let us give the difinition of quasi-periodic motions which will be necessary for the statements of our theorems.

Definition 5. Let $\boldsymbol{T}^n=\boldsymbol{R}^n/\boldsymbol{Z}^n=\{(x_1, \cdots, x_n); x_i \in \boldsymbol{R}, \text{ mod } 1, i=1, \cdots, n\}$ be the n-dimensional torus with usual Lebesgue measure, $dm=dx_1\cdot, \cdots, \cdot dx_n$. *Jacobi flow* with frequencies $\omega_1, \cdots, \omega_n$ is a classical flow $(\boldsymbol{T}^n, m, \tau_t)$, where (τ_t) is the one-parameter group of transformations defined by:

$$\tau_t x_i = x_i+\omega_i t, \text{ mod } 1, (i=1, \cdots, n).$$

It is easy to prove the following

Proposition 3. *An orbit of Jacobi flow with frequencies $\omega_1, \cdots, \omega_n$ is everywhere dense on $\boldsymbol{T}^n$, if and only if $\omega_1, \cdots, \omega_n$ are linearly independent over $\boldsymbol{Z}$. In this case, we call this Jacobi flow a quasi-periodic motion.*

§3. Main theorems. We state the theorems without proofs. The detailed proofs will be published in [3].

Theorem 1. *Let (M, μ, φ_t) be a classical ergodic flow and M be compact. If $\lambda_1, \cdots, \lambda_r \in \Lambda^\rho$ $(\rho \geqq 1)$ are linearly independent over $\boldsymbol{Z}$, then we can consider M as the total space of a locally trivial $C^{\rho-1}$-smooth fibre space over an r-dimensional torus $\boldsymbol{T}^r$, whose fibres are C^ρ-submanifolds. The flow (φ_t) is fibre preserving and the flow which is naturally induced on the base space $\boldsymbol{T}^r$ is a quasi-periodic motion with frequencies $\lambda_1, \cdots, \lambda_r$.*

Corollary. *Under the assumptions of Theorem* 1, *the group Λ^ρ*

is finitely generated and rank of $\Lambda^\rho \leq$ dimension of M.

Theorem 2. *Let (M, μ, φ_t) be classical ergodic flow and M be compact. If*

$$\sum_{\lambda \in \Lambda^\rho} \oplus H^{(\lambda)} = H \quad (\rho \geqq 1),$$

then (M, μ, φ_t) is C^ρ-isomorphic to a quasi-periodic motion as classical flows, i.e.,

$$(M, \mu, \varphi_t) \underset{C^\rho}{\simeq} (T^n, m, \tau_t) \quad (n = \dim M).$$

§4. Equi-continuous flows and some remarks. The author does not know whether the assumption of differentiability of eigenfunctions can be replaced by the assumption of continuity or not, but it is very likely that it can. We will show the partial solution for it, though it is far from the complete solution.

In this connection, we remark that if this conjecture is verified, then we can prove the following interesting theorem by means of this result and a theorem due to von Neumann and Halmos ([5] p. 349 Theorem 6), although this theorem can be proved with the help of theory of Lie groups.

Theorem 3 (*von Neumann-Halmos*). *Let (M, μ, φ_t) be a classical ergodic flow. If we can define the metric $d(x, y)$, compatible with the original topology of M, for which M is complete and the flow (φ_t) is equi-continuous with respect to t, i.e., $\forall \varepsilon > 0$, $\exists \delta > 0 : d(x, y) < \delta$ implies $d(\varphi_t x, \varphi_t y) < \varepsilon$ for all t, then (M, μ, φ_t) is C^0-isomorphic to a quasi-periodic motion (T^n, m, τ_t) as classical flows:*

$$(M, \mu, \varphi_t) \underset{C^0}{\simeq} (T^n, m, \tau_t) \quad (n = \dim M).$$

With the help of this theorem, we can prove the following

Theorem 4. *Let (M, μ, φ_t) be classical ergodic flow and M be compact. If*

$$\sum_{\lambda \in \Lambda^0} \oplus H^{(\lambda)} = H,$$

and, for any $x, y \in M$, $x \neq y$, there exists such continuous eigenfunction $f_\lambda(x) \in H^{(\lambda)}$ that $f(x) \neq f(y)$, then (M, μ, φ_t) is C^0-isomorphic to a quasi-periodic motion (T^n, m, τ_t) as classical flows:

$$(M, \mu, \varphi_t) \underset{C^0}{\simeq} (T^n, m, \tau_t) \quad (n = \dim M).$$

References

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