59. *A Geometric Condition for Smoothability of Bounded Combinatorial Manifold*

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1. Introduction. If we modify the paper [3] as follows, then Smoothability Theorem of that paper can be extended the case of bounded combinatorial manifold. For general terminology and definition, see [3].

Let M be a compact bounded combinatorial n-manifold piecewise linearly imbedded in a combinatorial $(n+k)$-manifold W^{n+k} without boundary and X, Y, Z be simplicial divisions of $M, \partial M, W$ such that X and Y are subcomplexes of Z and X respectively. Then $N(X, Z) \bmod Y$ denotes the star neighborhood of X in $Z \bmod Y$, that is, the polyhedron consists of simplices of Z containing simplices whose interior is contained in $|X-Y|$.

Definition 1. Let M be a compact bounded n-manifold imbedded piecewise linearly in euclidean $(n+k)$-space R, $k\geqq 1$. We say that M is *in smoothable position* in R if the following is satisfied.

Let K_0 and L_0 be simplicial divisions of M and R respectively, where K_0 is a complete subcomplex of L_0. And let H_0 be simplicial division of ∂M, where H_0 is a complete subcomplex of K_0.

Then there exist piecewise linear proper imbeddings

$$\varphi_i : M_i \to \partial(N(K_i', L_i') \bmod H_i') - \mathrm{Int}\, N(H_i', \partial(N(K_i', L_i') \bmod H_i')),$$

for each $0\leqq i\leqq k-1$, where $M_0=M$ and for $1\leqq i\leqq k$, $M_i=\varphi_{i-1}(M_{i-1})$ and where K_i, H_i, and L_i are simplicial subdivisions of $M_i, \partial M_i$ and $\partial(N(K_{i-1}', L_{i-1}') \bmod H_{i-1}') - \mathrm{Int}\, N(H_{i-1}', \partial(N(K_{i-1}', L_{i-1}') \bmod H_{i-1}'))$.

In the text, however, W_i stands for

$$\partial(N(K_{i-1}', L_{i-1}') \bmod H_{i-1}') - \mathrm{Int}\, N(H_{i-1}', \partial(N(K_{i-1}', L_{i-1}') \bmod H_{i-1}'))$$

and L_i will be the subcomplex of L_{i-1}' covering W_i for each $1\leqq i\leqq k$.

Then $\partial W_i = \partial N(H_{i-1}', \partial(N(K_{i-1}', L_{i-1}') \bmod H_{i-1}'))$.

Note that M_i is a combinatorial n-manifold with boundary, which is combinatorially equivalent to M, and W_i is a combinatorial $(n+k-i)$-manifold with boundary, for each $1\leqq i\leqq k$, satisfying $M_i \subset W_i$ and $W_1 \supset W_2 \supset \cdots \supset W_k$. Furthermore $N(K_0', L_0') \bmod H_0'$ is a regular neighborhood of $M \bmod \partial M$ in R^{n+k} and $N(K_i', L_i') \bmod H_i'$ is a regular neighborhood of $M_i \bmod \partial M_i$ in W_i in the sense of [1], $i=1$.

The extended result of [3] is the following.

Theorem 1. *If a compact bounded combinatorial n-manifold M is in smoothable position in $(n+k)$-space R, $k \geqq 1$, then M is smoothable.*

Theorem 2. *If the regular neighborhood U of M mod ∂M in R is combinatorially equivalent to $M \times B^k$ where B^k is a combinatorial n-ball, then M is smoothable.*

Proof. Using the uniqueness theorem for relative regular neighborhood [4 p. 21 Theorem 4.9] or [2], $U = N(K_0', L_0')$ mod H_0' keeping M fixed where K_0, L_0, H_0 are similar to Definition 1. Hence $N(K_0', L_0')$ mod $H_0' = M \times B^k$ and by Theorem 1 it is sufficient to show that M is smoothable position in R under the above condition. Since $N(K_0', L_0')$ mod $H_0' = M \times B^{k-1}$ there is a proper imbedding

$$\varphi_0 : M \to W_1 = M \times S^{k-1}$$

defined by taking $\varphi_0(x) = (x, x_0)$ for $x \in M_0$ where $S^{k-1} = \partial B^k$ is a combinatorial $(k-1)$-sphere and x_0 is a fixed point of S^{k-1}.

Let B^{k-1} be a combinatorial $(k-1)$-ball of S^{k-1} containing x_0 in the interior.

It is clear that $M \times B^{k-1}$ is a regular neighborhood of $M \times \{x_0\} (= M_1)$ mod ∂M_1 in $M \times S^{k-1} (= W_1)$ and since there exists a subdivision L_1 of $M \times S^{k-1}$ such that $M \times B^{k-1}$, $N(K_1', L_1')$ mod H_1' are satisfying the condition of [4, Theorem 4.9], $N(K_1', L_1')$ mod $H_1' = M \times B^{k-1}$ and there is a proper imbedding

$$\varphi_1 : M_1 \to W_2 \cong M \times S^{k-2}$$

defined by taking $y_1(y) = (y, y_0)$ for $y \in M$ where $S^{k-2} = \partial B^{k-1}$ and y_0 is a fixed point of S^{k-2}, and so on. Hence M is smoothable position in R, and therefore Theorem 2 is proved.

Suppose that a combinatorial n-manifold M is in smoothable position in R. Using Definition 1, M_k is combinatorially equivalent to M. Therefore Theorem 1 follows from Theorem 3 below in accordance with [7, p. 159].

Theorem 3. *Let a compact bounded n-manifold M be in smoothable position in euclidean $(n+k)$-space, $k \geqq 1$. Then M_k admits a transverse k-plane field over M_k.*

Theorem 4. *If the n-ball B^n is piecewise linearly imbedded in $(n+k)$-space R, $k \geqq 2$, then it is arbitrarily approximated by the n-ball which is in smoothable position.*

Since proof of Theorem 3 is completely analogous to [3], it is omitted. In the following we prove Theorem 4.

2. Proof of Theorem 4. Let B^n be an n-ball piecewise linearly and locally flatly imbedded in R^{n+k}, then there exist simplicial divisions K, L of B^n, R^{n+k} respectively such that $(N(K', L')$ mod $H', B^n)$ is an unknotted ball pair (B^{n+k}, B^n) where H is a simplicial division of ∂B compatible with K [1, Corollary 10].

In fact if a pair is unknotted then it is locally flat, because we triangulate with a standard pair.

Since (B^{n+k}, B^n) is an unknotted ball pair, there exists a *PL*-homeomorphism

$$h: (B^{n+k}, B^n) \to (I^{n+k}, I^n)$$

where $I=(-1, 1)$ and where I^i is imbedded in I^{i+1} as $I^i \times 0$ for $0 \leqq i \leqq n+k-1$. Hence $N(K', L') \bmod H' \cong B^n \times B^k$ therefore B^n is in smoothable position in R^{n+k} by Theorem 2.

On the other hand after Zeeman [8] locally knotting can not occur in codimension greater than 3.

Furthermore by [5, Corollary 1] any locally knotted proper embedding $f: B^n \to B^{n+2}$ is arbitrarily approximated by a locally flat embedding.

Therefore by the above remark we obtain the result.

References

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