54. A Remark on the Theorem of Bishop

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1. On normality of a family of pure-dimensional analytic sets in a domain of C^n, the following theorem of Oka [4] is well-known.

Theorem of Oka. *Let F be a family of pure-dimensional analytic sets in a domain of C^n. Then F is analytically normal if and only if the volumes of elements of F are locally uniformly bounded.*

This theorem was proved by T. Nishino [3] in the case of two variables. The proof of this theorem in the case of n variables was given in our former paper (Watanabe [6]).

On the other hand, the concept of geometric convergence was introduced by E. Bishop as follows.

Let $\{S_\nu\}$ be a sequence of closed subsets in a domain of C^n. It is said that $\{S_\nu\}$ converges geometrically to a closed set S if for any compact set K, $\{S_\nu \cap K\}$ is a convergent sequence in Comp(K)[1] and $S=\bigcup_K \lim(S_\nu \cap K)$ where K ranges over the compact sets. Further Bishop [1] proved the following.

Theorem of Bishop. *Let $\{S_\nu\}$ be a sequence of purely λ-dimensional analytic sets in a domain D of C^n. Suppose that $\{S_\nu\}$ converges geometrically to a closed set S in D. If the volumes of S_ν are uniformly bounded, then S is also an analytic set in D.*

We shall prove that in the above theorem of Bishop, S is also purely λ-dimensional if S is not empty.

2. Let $D=\Delta\times\{|w|<R\}$ be a domain of C^{n+1}, where Δ is a domain of $(z_1,\cdots,z_n)$-space $C^n(z)$. Then the following proposition is well-known (for example, Fujita [2]).

Proposition. *Let S be a purely λ-dimensional analytic set in D. Assume that S is contained in $\Delta\times\{|w|<R_0\}$ for some positive number $R_0<R$. Then the projection of S on Δ is also purely λ-dimensional analytic set in Δ.*

It follows from this:

Corollary. *Let $D=\Delta\times\{|w_1|<R\}\times\cdots\times\{|w_\mu|<R\}$ be a domain of $C^{\lambda+\mu}$ and S be a purely λ-dimensional analytic set in D. If S is contained in $\Delta\times\{|w_1|<R_0\}\times\cdots\times\{|w_\mu|<R_0\}$ for some positive number $R_0<R$, then $\mathfrak{A}=(S,\pi,\Delta)$ is an analytic cover, where π is a projection.*

1) For a definition of Comp (K), see [5].

Now let us prove the following

Theorem. *Let $\{S_\nu\}$ be a sequence of purely λ-dimensional analytic sets in a domain of $\boldsymbol{C}^n$. If $\{S_\nu\}$ converges geometrically to a non-empty analytic set S, then the local dimension $\dim_p S$ at each point $p \in S$ is at least λ.*

Proof. Suppose that there holds $\dim_p S = k < \lambda$ for some point $p \in S$. For simplicity we may assume that p is the origin. After a suitable change of the coordinate system, we can choose $(n-k)$ pseudo-polynomials $P_{k+1}(z';\zeta), P_{k+2}(z';\zeta)\cdots, P_n(z';\zeta)$, whose coefficients are holomorphic functions of $z'=(z_1,\cdots,z_k)$, and positive numbers ε_i $(i=1,2,\cdots,n)$ such that

(i) $S \cap U \subset S^* = \{(z', z_{k+1}, \cdots, z_n)\,;\, P_l(z';z_l)=0,\ l=k+1,\cdots,n\}$ for a neighbourhood U of the origin.

(ii) the roots of $P_l(z';\zeta)=0$ are all contained in the disc $|\zeta|<\varepsilon_l$ $(l=k+1,\cdots,n)$ for $z'=(z_1,\cdots,z_k)$ with $|z_j|\leqq\varepsilon_j$ $(j=1,2,\cdots,k)$. For sufficiently small positive numbers r,ρ, the polydisc $\Omega=\Delta\times\{|z_{k+1}|<\rho\}\times\cdots\times\{|z_n|<\rho\}$ is a relatively compact subset of U, where $\Delta=\{z'\,;\,|z_j|<r,\quad j=1,2,\cdots,k\}$.

We may assume that for $z'\in\Delta$, the roots of $P_l(z';\zeta)=0$ are all contained in the disc $|\zeta|<\dfrac{\rho}{2}=\rho'$. Then S^* is an analytic set in Ω and $S^*\cap\Omega\subset\Delta\times\{|z_{k+1}|<\rho'\}\times\cdots\times\{|z_n|<\rho'\}$. Since $\{S_\nu\}$ converges geometrically to S, it follows that $\lim\limits_{\nu}(S_\nu\cap\bar{\Omega})\subset S\cap\bar{\Omega}$, and it is easily seen that for a positive number ρ'' $(\rho'<\rho''<\rho)$ there is a positive integer ν_0 such that $S_\nu\cap\Omega$ is contained in $\Delta\times\{|z_{k+1}|<\rho''\}\times\cdots\times\{|z_n|<\rho''\}$ for $\nu\geqq\nu_0$. On the other hand, denoting $\Delta\times\{|z_{k+1}|<\rho\}\times\cdots\times\{|z_\lambda|<\rho\}$ by $\tilde{\Delta}$, we have $\Omega=\tilde{\Delta}\times\{|z_{\lambda+1}|<\rho\}\times\cdots\times\{|z_n|<\rho\}$ and so $S_\nu\cap\Omega\subset\tilde{\Delta}\times\{|z_{\lambda+1}|<\rho''\}\times\cdots\times\{|z_n|<\rho''\}$. But from the corollary of Proposition, the projection of $S_\nu\cap\Omega$ on the $(z_1,z_2,\cdots,z_\lambda)$-space is $\tilde{\Delta}$. This contradicts the fact that $S_\nu\cap\Omega$ is contained in $\Delta\times\{|z_{k+1}|<\rho''\}\times\cdots\cdots\times\{|z_n|<\rho''\}$. Q.E.D.

On the other hand, from the estimation of the Hausdorff measure and the relation between the volume and the Hausdorff measure, the dimension of S is at most λ if a sequence of purely λ-dimensional analytic sets converges geometrically to an analytic set S and if the 2λ-dimensional volumes of S_ν are uniformly bounded ([5]). Therefore we have from our theorem, the following

Corollary. *In the theorem of Bishop, if the limit set S is not empty, then S is also purely λ-dimensional.*

3. Here we shall give some properties of geometric and analytic convergence.

Let $\{S_\nu\}$ be a sequence of closed sets in a domain D of $\boldsymbol{C}^n$. Suppose that $\{S_\nu\}$ converges geometrically to a closed set S in D. Since $\lim_\nu (S_\nu \cap K_\mu) \subset \lim_\nu (S_\nu \cap K_{\mu+1})$, we have $S \subset \lim_\mu \lim_\nu (S_\nu \cap K_\mu)$ for every sequence $\{K_\mu\}$ of compact sets such that $K_\mu \subset K_{\mu+1} \cdots$, and $D = \bigcup_\mu K_\mu$. On the other hand, we have $S \supset \lim_\mu \lim_\nu (S_\nu \cap K_\mu)$ from the very definition, and hence $S = \lim_\mu \lim_\nu (S_\nu \cap K_\mu)$. The converse is not necessarily true as the following example shows.

Example. Let $D = \{(z_1, z_2) \in C^2 ; |z_1 - 1| < 1, |z_2| < 1\}$ be a domain of C^2 and $\{S_n\}$ be a sequence of analytic sets such that $S_1 = \left\{(z_1, z_2) \in D ; z_1 = \frac{1}{2}\right\}$, $S_2 = \left\{(z_1, z_2) \in D ; z_1 = 1 + \frac{1}{2}\right\}$, $\cdots$, $S_{2n} = \left\{(z_1, z_2) \in D ; z_1 = 1 + \frac{1}{2^n}\right\}$, $S_{2n+1} = \left\{(z_1, z_2) \in D ; z_1 = \frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{2^n}\right\}$, $\cdots$. Then since $S_{2n} \cap K \neq \emptyset$, and $S_{2n+1} \cap K = \emptyset$ for the compact set $K = \left\{(z_1, z_2) \in D ; \left|z_1 - \frac{5}{4}\right| \leqq \frac{1}{4}, |z_2| \leqq \frac{1}{2}\right\}$, $\{S_n\}$ does not converge geometrically. But it is obvious that $\lim_\mu \lim_\nu (S_\nu \cap K_\mu) = \{(z_1, z_2) \in D ; z_1 = 1\}$ for every exhaustion of D by compact sets K_μ. However if $\lim_\nu (S_\nu \cap K_\mu)$ exists for sufficiently large μ, then there is a subsequence $\{S_{\nu_j}\}$ of $\{S_\nu\}$ which converges geometrically to a closed set S^* and $\lim_\nu (S_\nu \cap K_\mu) = \lim_j (S_{\nu_j} \cap K_\mu)$ for sufficiently large μ, and hence we have $S^* = \lim_\mu \lim_j (S_{\nu_j} \cap K_\mu) = \lim_\mu \lim_j (S_\nu \cap K_\mu)$ for sufficiently large μ.

Summing up the above result, we have

Proposition 1. *If $\{S_\nu\}$ converges geometrically to a closed set S in D, then it holds that $S = \lim_\mu \lim_\nu (S_\nu \cap K_\mu)$ for every sequence $\{K_\mu\}$ of compact sets such that $K_\mu \subset K_{\mu+1} \cdots$, and $D = \bigcup_\mu K_\mu$. Further if $S = \lim_\mu \lim_\nu (S_\nu \cap K_\mu)$ exists, then S is closed and there is a subsequence of $\{S_\nu\}$ which converges geometrically to S.*

Next we consider the case of analytic convergence of a sequence of pure-dimensional analytic sets.

Suppose that a sequence $\{S_\nu\}$ of pure-dimensional analytic sets in a domain D of C^n converges analytically to S.[2] We shall show that $S \cap K = \lim_\nu (S_\nu \cap K)$ for a compact set K such that $S \cap \mathring{K} \neq \emptyset$.

Let $S^{(\varepsilon)} = \bigcup_{z' \in S} \{z ; \rho(z, z') < \varepsilon\}$, where $\rho(z, z')$ means the Euclid distance between z and z'. If $S_{\nu_j} \cap K - S^{(\varepsilon)} \cap K \neq \emptyset$ for a sequence of positive integers $\nu_1 < \nu_2 < \cdots$, then we can choose a sequence of points $p_{\nu_j} \in S_{\nu_j} \cap K - S^{(\varepsilon)} \cap K$. Since K is compact it may be assumed that $p_{\nu_j} \to p$. From the assumption p is not contained in S. On the other

2) For a definition of analytic convergence, see [2], [6].

hand, from the definition of analytic convergence there are a neighbourhood U of p and holomorphic functions $f_k^{(\nu)}$, $k=1,2,\cdots,l$ in U such that $S_\nu \cap U=\{z\in U; f_k^{(\nu)}(z)=0,\ k=1,2,\cdots,l\}$.

Moreover, since $f_k^{(\nu)}(z)$ converges uniformly to a holomorphic function $f_k^{(0)}(z)$ in U, $f_k^{(\nu_j)}(z)$ also converges uniformly to $f_k^{(0)}(z)$ in U. We have $|f_k^{(0)}(p)|=\delta_k>0$ since p is not contained in S. But since $f_k^{(\nu_j)}(z)$ converges uniformly to $f_k^{(0)}(z)$, it holds $|f_k^{(0)}(p)|\leqq|f_k^{(0)}(p)-f_k^{(0)}(p_{\nu_j})|+|f_k^{(0)}(p_{\nu_j})-f_k^{(\nu_j)}(p_{\nu_j})|<\delta_k$ for sufficiently large j. This is a contradiction and hence we have $S_{\nu_j}\cap K-S^{(\varepsilon)}\cap K=\emptyset$ for sufficiently large j.

Thus there is a positive integer ν_0 depending only on ε such that $S_\nu\cap K\subset S^{(\varepsilon)}\cap K$ for $\nu\geqq\nu_0$. This means that $\{S_\nu\cap K\}$ converges to $S\cap K$ in Comp(K). Thus we have

Proposition 2. *If a sequence $\{S_\nu\}$ of pure-dimensional analytic sets converges analytically to S, then it holds that $S=\lim_\mu \lim_\nu (S_\nu\cap K)$ for every sequence $\{K_\mu\}$ of compact sets such that $K_\mu\subset K_{\mu+1}\cdots$, and $D=\bigcup_\mu K_\mu$.*

Remark. Even if $\lim_\nu (S_\nu\cap K_\mu)$ exists, the sheet numbers of S_ν need not be bounded. Hence we can not always choose a subsequence of $\{S_\nu\}$ which converges analytically. Such an example was given in former paper ([6]).

References

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