85. *On Certain Mixed Problem for Hyperbolic Equations of Higher Order.* II

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(Comm. by Kinjirô KUNUGI, M. J. A., May 12, 1969)

1. Introduction and results. In the present note, we will extend our results stated before ([3]).

Let Ω be a domain with a bounded boundary Γ of $\boldsymbol{R}^n$. Here we consider a strongly hyperbolic equation

$$(1)\qquad Lu=\left(\frac{\partial^{2m}}{\partial t^{2m}}+a_1(x, D)\frac{\partial^{2m-1}}{\partial t^{2m-1}}+\cdots+a_{2m}(x, D)\right)u$$
$$+(\text{lower order terms})u=f,$$
$$a_k(x, D)=\sum_{|\alpha|=k}a_\alpha(x)D^\alpha,\ D_j=\frac{1}{\sqrt{-1}}\frac{\partial}{\partial xj},$$

and let all of the roots $\tau_i(x, \xi)$ $(i=1, 2, \cdots, 2m)$ with respect to τ of the equation $\tau^{2m}+a_1(x, \xi)\tau^{2m-1}+\cdots+a_{2m}(x, \xi)=0$ be pure imaginary and distinct mutually, not zero uniformly for $x\in\bar{\Omega}$, $|\xi|=1$.

Here we assume that, after applying any coordinate transformation $(U\cap\Omega, \Gamma\cap\Omega)\mapsto(\boldsymbol{R}^n_+=\{y\in\boldsymbol{R}^n\,|\,y_n>0\}, \{y\,|\,y_n=0\})$ such that on the boundary the conormal direction of a given uniformly strongly elliptic operator $a(x, D)$ of order 2 is changed into the normal direction, the coefficients of the principal part of (1) containing odd power of $\dfrac{\partial}{\partial y_n}$ are zero on the boundary $y_n=0$.

Then we obtain the following

Theorem. *For any* $f(t, x)\in C^1([0, T], L^2(\Omega))$ *and for any initial conditions* $\left(u(0, x), \dfrac{\partial u}{\partial t}(0, x), \cdots, \dfrac{\partial^{2m-1}u}{\partial t^{2m-1}}(0, x)\right)\in D(a^m)\times\cdots\times D(a^{\frac{1}{2}})$, *there exists a unique solution* u *of* (1), *satisfying boundary conditions such that* $\left(u(t, x), \dfrac{\partial u}{\partial t}(t, x), \cdots, \dfrac{\partial^{2m}u}{\partial t^{2m}}(t, x)\in C^\circ([0, T], D(a^m)\times D(a^{m-\frac{1}{2}})\times\cdots\times D(a^{\frac{1}{2}})\times L^2(\Omega)\right)$. *Here* $D(a)=H^2(\Omega)\cap H^1_0(\Omega)$ *or* $\left\{u\in H^2(\Omega)\,\middle|\,\left(\dfrac{\partial}{\partial n}+\rho(x)\right)u\Big|_\Gamma=0\right\}$. *Furthermore* $\dfrac{\partial}{\partial n}$ *is the conormal derivative of* a *and* $\rho(x)\in C^\infty(\Gamma)$.

To prove the theorem above mentioned, we need to extend our singular integral operators defined on $\boldsymbol{R}^n_+$ to ones defined over Ω ([11]).

For simplicity of descriptions we assume that all coefficients and Γ are sufficiently smooth, but the conditions with respect to the regularities of these shall be weakened as in the Cauchy problems of hyperbolic equations. Furthermore we will use the notations and definitions described in the previous paper ([3]).

The detailed treatment and other interesting results shall be published elsewhere.

2. Singular integral operators defined over Ω with boundary conditions.

Lemma 1. *There exist a finite covering $\{U_\alpha\}$ of $\bar{\Omega}$ and diffeomorphisms T_α from U_α into $\boldsymbol{R}^n$ such that the following* 1), 2) *and* 3) *are satisfied*:

1) *If $U_\alpha \cap \Gamma \neq \phi$, then $T_\alpha(U_\alpha \cap \Omega) \subset \boldsymbol{R}^n_+$, $T_\alpha(U_\alpha \cap \Gamma) \subset \bar{\boldsymbol{R}}^n_+ - \boldsymbol{R}^n_+$, and $T_\alpha(U_\alpha \cap \Gamma)$ contains origin.*

2) *Let $a_0(x, D)$ be the principal part of $a(x, D)$ and let $b_\alpha(y, \xi) = \sum_{i,j} b^{(\alpha)}_{ij}(y)\xi_i\xi_j = a_0(T^{-1}_\alpha(y), (dT^*_\alpha)(\xi))$ for $y \in T_\alpha(U_\alpha)$, $\xi \in \boldsymbol{R}^n$, then $b^{(\alpha)}_{nj}(y) = 0$ for $y_n = 0$ and for $n \neq j$.*

3) *If $U_\alpha \cap U_\beta \cap \Gamma \neq \phi$ and $y \in T_\alpha(U_\alpha \cap U_\beta)$, then the n-th component of $T_\beta(T^{-1}_\alpha(y))$ is equal to y_n.*

Let $\{\varphi_i\}$ be a partition of unity with respect to $\bar{\Omega}$ such that $\varphi_i u \in D(a)$ for $u \in D(a)$, and $\{\mathrm{Supp}\,\varphi_i\}$ is a star-finite refinement of $\{U_\alpha\}$.

Definition 1. We denote by $\bar{\mathfrak{A}}_\Omega$ the set of $\sigma(x, \xi)$ such that $\sigma(x, \xi) \in \Xi^\infty_4(\bar{\Omega} \times (\boldsymbol{R}^n - \{0\}))$ and for $\mathrm{Supp}\,\varphi_i \cap \mathrm{Supp}\,\varphi_j \cap \Gamma \neq \phi$

$$(T_{\alpha*}\circ\varphi_i)(\sigma(T^{-1}_\alpha(y), (dT^*_\alpha)(\xi))\circ T_{\alpha*}\circ\varphi_j \in \bar{\mathfrak{A}},$$

where $\bar{\mathfrak{A}}$ is the set of all $\sigma(x, \xi) = \sum_{i=1}^\infty \sigma_i(x, \xi)$ for $\sigma_i(x, \xi) \in \mathfrak{A}$ and $\sum_{i=1}^\infty M_s(\sigma_i) < \infty$ for any integer $s (\geqslant 0)$. (For the definition of Ξ^∞_4 and $\mathfrak{A}$, see [3]).

Definition 2. For $\sigma(x, \xi) \in \bar{\mathfrak{A}}_\Omega$ we associate with the singular integral operator $\sigma(x, D)$ as follows: For $u \in L^2(\Omega)$

$$\sigma(x, D)u = \sum_{i,j} \sigma_{ij}(x, D)u,$$

where we define $\sigma_{ij}(x, D)u$ as the ordinary singular integral operator if $\mathrm{Supp}\,\varphi_j \cap \Gamma = \phi$, otherwise

$$\begin{aligned}\sigma_{ij}(x, D)u &= T^*_\alpha((T_{\alpha*}\circ\varphi_i)\cdot\tilde{\sigma}(T^{-1}_\alpha(y), (dT^*_\alpha)(D_y))\circ T_{\alpha*}\circ\varphi_j\circ u) \quad \text{if} \\ &\qquad \Gamma \cap \mathrm{Supp}\,\varphi_j \cap \mathrm{Supp}\,\varphi_i \neq \phi \quad \text{and} \quad U_\alpha \supset \mathrm{Supp}\,\varphi_i \cup \mathrm{Supp}\,\varphi_j, \\ &= 0 \quad \text{if} \quad \mathrm{Supp}\,\varphi_j \cap \mathrm{Supp}\,\varphi_i = \phi.\end{aligned}$$

In the above $\tilde{\sigma}(T^{-1}_\alpha(y), (dT^*_\alpha(\xi)) = \psi(y)\sigma(T^{-1}_\alpha(y), (dT^*_\alpha)(\xi)) + (1 - \psi(y)) \cdot \sigma(T^{-1}_\alpha(0), (dT^*_\alpha)(\xi))$ for $\psi(y) \in C^\infty_0(T_\alpha(U_\alpha))$ such that $\psi(y) = 1$ for $y \in T_\alpha(\mathrm{Supp}\,\varphi_i \cup \mathrm{Supp}\,\varphi_j))$.

Definition 3. We define the operator $\tilde{\Lambda}_\pm$ as in Definition 2 such that

$$\tilde{\Lambda}_{\pm}u=\sum_{i,j} T^*_{\alpha}\circ(T_{\alpha *}\circ\varphi_i)\sqrt{b_{\alpha}}(y, D)\Lambda_{\pm}\circ T_{\alpha *}\circ(\varphi_j\circ u) \text{ for any } u\in D(a^{\frac{1}{2}}).$$

Here we remark that $\sqrt{b_{\alpha}}\left(y, \frac{\xi}{|\xi|}\right)\in\overline{\mathfrak{A}}$ if $U_{\alpha}\cap\Gamma\neq\phi$.

Then we obtain following Lemmas.

Lemma 2. *For $\sigma_1(x, \xi)$ and $\sigma_2(x, \xi)\in\overline{\mathfrak{A}}_{\Omega}$ the operators*

$$\sigma_1(x, D)\tilde{\Lambda}_{\pm}-\tilde{\Lambda}_{\pm}\sigma_1(x, D),\ (\sigma_1(x, D)^*-\sigma_1^{\#}(x, D))\tilde{\Lambda}_{\pm} \quad \text{and}$$
$$((\sigma_1\circ\sigma_2)(x, D)-\sigma_1(x, D)\sigma_2(x, D))\tilde{\Lambda}_{\pm}$$

are extended to operators of $B(L^2(\Omega), L^2(\Omega))$.

Lemma 3. *For sufficiently large $k>0$ we have the following estimate: There exists a constant $c>0$*

$$\|(\tilde{\Lambda}_{\pm}+k)u\|_{L^2(\Omega)}\geqslant c\|u\|_{1,L^2(\Omega)}$$

for $u\in D(a^{\frac{1}{2}})$.

Lemma 4. *Let $\sigma(x, \xi)\in\overline{\mathfrak{A}}_{\Omega}$ with $\sigma(x, \xi)\geqslant c_1>0$ for $x\in\overline{\Omega}$, $|\xi|=1$. Then we have the following estimate: There exists a constant $c>0$*

$$\|(\sigma(x, D)\tilde{\Lambda}_{\pm}+k)u\|^2_{L^2(\Omega)}\geqslant c(\|\tilde{\Lambda}_{\pm}u\|^2_{L^2(\Omega)}+\|u\|^2_{L^2(\Omega)})$$

for $u\in D(a^{\frac{1}{2}})$.

Here we remark that our definitions depend on the choice of $\{U_{\alpha}\}$, $\{T_{\alpha}\}$ and $\{\varphi_j\}$, but we can prove that our singular integral operators are invariant under coordinate transformations with the properties in Lemma 1, in the same sense as ones used by Seeley ([11]).

3. To prove Theorem, we need the following

Lemma 5. *If $a_k(x, D)$ satisfies the assumption stated in § 1, then we find that $a_{2k+1}(x, D)u=\sigma_{2k+1}(x, D)\tilde{\Lambda}_{\pm}a^k u+$(lower order terms)$u$, $a_{2k}(x, D)u=\sigma_{2k}(x, D)a^k u+$(lower order terms)$u$ for $u\in D(a^{k+\frac{1}{2}})$ or $u\in D(a^k)$ $(k\geqslant 0)$ respectively, where $\sigma_k(x,\xi)=\dfrac{a_k(x, \xi)}{(a_0(x, \xi))^{k/2}}\in\overline{\mathfrak{A}}_{\Omega}$. Here we may consider $\rho(x)=0$ on Γ in the Neumann case.*

To prove Lemma 5, we first multiply u by φ_j, then using coordinate transformation T_{α} such that $U_{\alpha}\supset\operatorname{Supp}\varphi_j$ we replace $\dfrac{\partial^2}{\partial y_n^2}$ in the principal part of $a_k(T_{\alpha}^{-1}(y),(dT^*_{\alpha})(D_y))$ by $\dfrac{\partial^2}{\partial y_n^2}=b_{nn}^{-1}\Big((a(T_{\alpha}^{-1}(y), (dT^*_{\alpha})(D_y))$ $-\sum_{i,j=1}^{n-1} b_{ij}\dfrac{\partial^2}{\partial y_i\partial y_j}-\sum_{i=1}^{n-1} b_{in}\dfrac{\partial^2}{\partial y_i\partial y_n}+\text{(lower order terms)}\Big)$. Furthermore considering the replaced one in Ω, we transfer φ_j forward their terms. Then we have only to prove the following

Lemma 6. *Let $X_{ij}=\dfrac{\partial^2}{\partial y_i\partial y_j}$, $Y_i=y_n c_{in}(y)\dfrac{\partial^2}{\partial y_i\partial y_n}(i,j=n)$. Then every term $\sum_{|\beta|+|\gamma|=k} X^{\beta_1}_{i_1j_1}\cdots X^{\beta_p}_{i_pj_p}\cdot Y^{\gamma_1}_{k_1}\cdots Y^{\gamma_q}_{k_q}\circ(T_{\alpha *}\circ\varphi_j\circ u)$ (Supp $\varphi_j\subset U_{\alpha}$) is represented as $\sigma(x, D)a^k u+$(lower order terms)u for $u\in D(a^k)$, considering it in Ω, here $\sigma(x, \xi)\in\overline{\mathfrak{A}}_{\Omega}$.*

Then writing $\tilde{a}=a+\lambda$, $\tilde{\Lambda}_{\pm,\lambda}=\tilde{\Lambda}_{\pm}+\lambda$, λ; a large constant, (1) is rewritten as

$$(2)\qquad \mathrm{Lu}=\left(\frac{\partial^{2m}}{\partial t^{2m}}+\alpha_1(x,D)\tilde{\Lambda}_{\pm,\lambda}\frac{\partial^{2m-1}}{\partial t^{2m-1}}+\alpha_2(x,D)\tilde{a}\frac{\partial^{2m-2}}{\partial t^{2m-2}}+\cdots\right.$$
$$\left.+\alpha_{2m-1}(x,D)\tilde{\Lambda}_{\pm,\lambda}\tilde{a}^{m-1}\frac{\partial}{\partial t}+\alpha_{2m}(x,D)\tilde{a}^m\right)u$$
$$+\left(\beta_1\frac{\partial^{2m-1}}{\partial t^{2m-1}}+\cdots+\beta_{2m}\right)u=f$$

for $\left(u, \frac{\partial u}{\partial t}, \cdots, \frac{\partial^{2m-1}u}{\partial t^{2m-1}}\right)\in D(a^m)\times D(a^{m-\frac{1}{2}})\times\cdots\times D(a^{\frac{1}{2}})$.

Hence setting

$$A=\begin{pmatrix} 0 & 1 & & & & \\ & & \ddots & & & \\ & 0 & & 1 & & \\ & & \ddots & & & \\ & & & 0 & 1 \\ -\alpha_{2m}\tilde{a}^m, & -\alpha_{2m-1}\tilde{\Lambda}_{\pm,\lambda}\tilde{a}^{m-1}, & \cdots, & -\alpha_2\tilde{a}, & -\alpha_1\tilde{\Lambda}_{\pm,\lambda} \end{pmatrix}$$

$$Q=\begin{pmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ -\beta_{2m}, & -\beta_{2m-1}, & \cdots, & -\beta_1 \end{pmatrix},\qquad U={}^t\left(u, \frac{\partial u}{\partial t}, \cdots, \frac{\partial^{2m-1}u}{\partial t^{2m-1}}\right)$$

$F={}^t(0, 0, \cdots, 0, f)$,

we can rewrite (2) as

$$(3)\qquad E\frac{dU}{dt}=(A+Q)U+F,\qquad E\text{: unit matrix.}$$

Here the domain we consider is $\mathfrak{G}(\tilde{\Lambda}_{\pm})=D(a^{m-\frac{1}{2}})\times D(a^{m-1})\times\cdots\times D(a^{\frac{1}{2}})\times L^2(\Omega)$, and the domain of A is $D(a^m)\times D(a^{m-\frac{1}{2}})\times\cdots\times D(a^{\frac{1}{2}})$. We define the norm: $\|u\|_{D(a^{j+\frac{1}{2}})}=\|\tilde{\Lambda}_{\pm,k}\tilde{a}^j u\|_{L^2(\Omega)}$ for $u\in D(a^{j+\frac{1}{2}})$, $\|u\|_{D(a^j)}=\|\tilde{a}^j u\|_{L^2(\Omega)}$ for $u\in D(a^j)$.

Similarly to Leray's method, for the roots $\tau_j(x,\xi)$ $(j=1,\cdots,2m)$ of the equation $\tau^{2m}+\frac{a_1(x,\xi)}{(a_0(x,\xi))^{\frac{1}{2}}}\tau^{2m-1}+\frac{a_2(x,\xi)}{(a_0(x,\xi))}\tau^{2m-2}+\cdots+\frac{a_{2m}(x,\xi)}{(a_0(x,\xi))^m}=0$, we set $\tau_j(x,\xi)=\sqrt{-1}\,\mu_j(x,\xi)$.

Furthermore we set

$$R(x,\xi)=\begin{pmatrix} 1 & \cdots & 1 \\ \mu_1(x,\xi) & \cdots & \mu_{2m}(x,\xi) \\ & \cdots & \\ (\mu_1(x,\xi))^{2m-1} & \cdots & (\mu_{2m}(x,\xi))^{2m-1} \end{pmatrix}\qquad S=RR^t,$$

and $\quad B(x,\xi)=S^{-1}\det S$.

Using it, we give an equivalent norm in $\mathfrak{G}(\tilde{\Lambda}_{\pm})$. For $U\in\mathfrak{G}(\tilde{\Lambda}_{\pm})$, we define $\|U\|_B$ such that

$$\|U\|_B^2=\mathrm{Re}(BE(i)E(\tilde{\Lambda}_{\pm})U, E(i)E(\tilde{\Lambda}_{\pm})U)_{L^2(\Omega)}+\beta\|\tilde{\Lambda}_{\pm,k}^{-1}E(\tilde{\Lambda}_{\pm})U\|_{L^2(\Omega)}^2,$$

here $E(\Lambda_\pm)=\begin{pmatrix} \tilde{\Lambda}_{\pm,\lambda}\tilde{a}^{m-1} & & & & & \\ & \tilde{a}^{m-1} & & & & \\ & & \ddots & & & \\ & & & \tilde{a} & & \\ & & & & \tilde{\Lambda}_{\pm,\lambda} & \\ & & & & & 1 \end{pmatrix}$, $E(i)=\begin{pmatrix} i^{2m-1} & & & & \\ & i^{2m-2} & & & \\ & & \ddots & & \\ & & & i & \\ & & & & 1 \end{pmatrix}$,

$i=\sqrt{-1}$,

β: a sufficiently large constant.

Using this norm, we find the following estimate: for $U \in D(A)$,

$$\|(\tau E-(A+Q)U\|_B \geqslant (\tau-\tau_0)\|U\|_B \quad \text{for} \quad \tau>\tau_0.$$

To show this estimate, we remark that

$$E(\tilde{\Lambda}_\pm)A=\begin{pmatrix} 0 & 1 & & & \\ & & \ddots & & \\ & & \cdot & 1 & \\ & & & \ddots & \\ & & & 0 & 1 \\ -\alpha_{2m}, & -\alpha_{2m-1}, & \cdots, & -\alpha_2, & -\alpha_1 \end{pmatrix}\tilde{\Lambda}_{\pm,k}E(\tilde{\Lambda}_\pm)$$

$$+(\text{system of bounded operators of } B(L^2(\Omega), L^2(\Omega))\cdot E(\tilde{\Lambda}_\pm)$$

$$\equiv P\tilde{\Lambda}_{\pm,\lambda}E(\tilde{\Lambda}_\pm)+P_1E(\tilde{\Lambda}_\pm),$$

and setting $P=iE(i)^{-1}P_2E(i)$, then we find that $B(x,\xi)P_2(x,\xi)$ is real symmetric.

Finally we see from Lemma 4 that

$$\begin{aligned} L&=\tau E-P\tilde{\Lambda}_{\pm,\lambda}+P_1+Q_1, \qquad E(\tilde{\Lambda}_\pm)Q=Q_1E(\tilde{\Lambda}_\pm),\\ &=\tau E-(P\tilde{\Lambda}_\pm+\Lambda P+P_1+Q_1)\\ &\equiv \tau E-(P\tilde{\Lambda}_\pm+Q_2) \end{aligned}$$

is a linear mapping from $(D(\tilde{\Lambda}_{\pm,\lambda}))^{2m}$ onto $(L^2(\Omega))^{2m}$.

By virtue of semi-group theory, we can prove Theorem.

Remark. In our proof of Theorem we can't use the local existence theorem to our problem, because local uniqueness theorem of that problem has not yet been proved in general and $\varphi_k u$ does not always belong to $D(a^l)$ $(l\geqslant 2)$ even if $u \in D(a^{l+j})$ $(j\geqslant 0)$.

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