# 103. *On the Global Solution of a Certain Nonlinear Partial Differential Equation*

By Masayoshi TSUTSUMI and Riichi IINO
Waseda University

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**1. Introduction.** We consider the following fourth order partial differential equation

$$\partial^2 y/\partial t^2=(1+\alpha(\partial y/\partial x)^{2p})\partial^2 y/\partial x^2-\beta\partial^4 y/\partial x^4, \tag{1}$$

where $\alpha$ and $\beta$ are positive constants and $p=1,2,\cdots$, which is deeply connected with the study of the anharmonic lattice (see [1]).

Here we consider the initial-boundary value problem for (1) with initial values

$$y(0,x)=f(x), \qquad \partial y/\partial t(0,x)=g(x), \tag{2}$$

and with periodic boundary condition

$$y(t,x)=y(t,x+1) \qquad \text{for all } x \text{ and } t. \tag{3}$$

Then we have the following theorem being concerned with the global solution for the problem:

**Theorem.** *For every $\alpha>0$, $\beta>0$, and for every real 1-periodic initial functions $f\in W_2^{(6)}(0,1)$, $g\in W_2^{(4)}(0,1)$, there exists the unique function which satisfies* (1), (2) *and* (3) *in the classical sense in the whole $(t,x)$ plane.*

The method of proof is the semi-discrete approximation similar to that presented by Sjöberg [2].

The authors were announced by Nisida [3] that he independently treated the same problem by means of the theory of semi-groups.

**2. Proof of existence of the global solution.** In order to prove the existence of the desired solution we employ the following semi-discrete approximation:

$$\begin{aligned} &d^2 y_N(t,x_r)/dt^2=D_+[D_-y_N(t,x_r)+\alpha(D_-y_N(t,x_r))^{2p+1}/2p+1] \\ &\qquad -\beta D_+^2D_-^2 y_N(t,x_r), \qquad r=1,2,\cdots,N \\ &y_N(0,x_r)=f(x_r), \qquad dy_N/dt(0,x_r)=g(x_r), \qquad r=1,2,\cdots,N, \\ &y_N(t,x_r)=y_N(t,x_{r+N}), \qquad r=1,2,\cdots,N \qquad \text{and all } t \end{aligned} \tag{4}$$

where the mesh-width $h=1/N$, $N$ natural number, $x_r=rh$ and the difference operators $D_+$ and $D_-$ are defined by

$$hD_+y(x_r)=y(x_{r+1})-y(x_r), \qquad hD_-y(x_r)=y(x_r)-y(x_{r-1}).$$

For every $h>0$ the solution of the problem (4) uniquely exists on the basis of the theory of ordinary differential equations. The solution $y_N(t,x_r)$, fixed $N$, is a grid-function defined for $x_r=rh$. We may write $y_N(t,x_r)=y_r(t)$ for the sake of simplicity.

We denote by $(f, g)$ the scalar product and by $\|f\|$ the norm in the space $L_2(0, 1)$, that is

$$(f, g)=\int_0^1 \overline{f(x)}g(x)dx \qquad \text{and} \qquad \|f\|^2=(f, f).$$

On the other hand, in the space of grid-functions we define

$$(f, g)_h=\sum_{r=1}^{N} \overline{f(x_r)}g(x_r)h \qquad \text{and} \qquad \|f\|_h^2=(f, f)_h.$$

Now we are going to write down a discrete analogue of Sobolev's theorem.

**Lemma 1.** *Let $\sigma$ and $\tau$ be integers with the property $0 \leq \tau < \sigma$ and $\sigma \leq N/2-1$. Then to every constant $\varepsilon>0$ there exists a constant $C(\varepsilon)>0$, independent of $N$-periodic grid-functions $u$ and $h$, such that*

$$\|D_+^{\tau_1}D_-^{\tau_2}u\|_h^2 \leq \max_{1\leq r\leq N}|D_+^{\tau_1}D_-^{\tau_2}u|^2 \leq \varepsilon\|D_+^{\sigma_1}D_-^{\sigma_2}u\|_h^2+C(\varepsilon)\|u\|_h^2, \tag{5}$$

*where $\sigma=\sigma_1+\sigma_2$, $\tau=\tau_1+\tau_2$, $\sigma_i, \tau_j \geq 0$, $i, j,=1, 2$.*

Here we define

$$E_1(t)=(\|dy/dt\|_h^2+\beta\|D_-^2 y\|_h^2+\alpha\|(D_-y)^{p+1}\|_h^2/(2p+1)(p+1) + \|D_-y\|_h^2)/2, \tag{6}$$

$$E_2(t)=(\|dv/dt\|_h^2+\beta\|D_-^2 v\|_h^2+\|D_-v\|_h^2)/2, \tag{7}$$

$$E_3(t)=(\|dw/dt\|_h^2+\beta\|D_-^2 w\|_h^2+\|D_-w\|_h^2)/2, \tag{8}$$

where $v_r=dy_r(t)/dt$, $w_r=d^2y_r(t)/dt^2$. Then we obtain the following:

**Lemma 2.** *For an arbitrary finite interval $0\leq t\leq T$, there exist constants $K_i$, $i=1, 2, 3$, which are independent of $h$, such that*

$$E_1(t)\leq K_1, \tag{9}$$

$$E_2(t)\leq K_2, \tag{10}$$

$$E_3(t)\leq K_3. \tag{11}$$

**Proof.** Differentiating (6) with respect to $t$, using the periodicity of the function $y_r$, and the system (4), we have

$$dE_1(t)/dt=0$$

which implies $E_1(t)=E_1(0)\leq K_1$.

In virtue of the following inequality

$$\|y(t)\|_h^2 \leq 2\left(t\int_0^t \|dy(s)/dt\|_h^2 ds + \|f\|_h^2\right),$$

and of (9), we get, for an arbitrary finite interval $0\leq t\leq T$,

$$\|y(t)\|_h^2 \leq k_1 \tag{12}$$

where $k_1$ is a constant independent of $h$.

Now the function $v_r(t)=dy_r(t)/dt$ satisfies the equation

$$d^2v_r(t)/dt^2=D_+D_-v_r+\alpha(D_+y_r)^{2p}D_+D_-v_r+\alpha D_-v_rD_+(D_-y_r)^{2p} - \beta D_+^2D_-^2 v_r \tag{13}$$

which is obtained by differentiating the equation (4) with respect to $t$. Differentiating (7), using the periodicity of the function $v_r(t)$ and the equation (13), we have

$$dE_2(t)/dt=\alpha(dv/dt, (D_+y)^{2p}D_+D_-v+D_-vD_+(D_-y)^{2p})_h.$$

Since

$$(dv/dt,\ (D_+y)^{2p}D_+D_-v)_h \le \max_r |D_+y_r|^{2p}(\|dv/dt\|_h^2 + \|D_+D_-v\|_h^2)/2,$$

$$\begin{aligned}(dv/dt,\ D_-vD_+(D_-y)^{2p})_h \\ \le 2p \max_r |D_-y_r|^{2p-1} \max_r |D_-v_r|\, \|dv/dt\|_h \|D_+D_-y\|_h \\ \le p \max_r |D_-y_r|^{2p-1} \|D_+D_-y\|_h(\|dv/dt\|_h^2 + \varepsilon\|D_-^2 v\|_h^2 + C(\varepsilon)\|D_-v\|_h^2),\end{aligned}$$

we obtain

$$dE_2(t)/dt \le k_2 E_2(t),$$

where $k_2$ is a constant independent of $h$, which implies

$$E_2(t) \le E_2(0) \exp k_2 T = K_2, \qquad 0 \le t \le T.$$

The inequality (11) may be driven in the similar way as (10).

(q.e.d.)

**Lemma 3.** *There exist constants $m_i$, $i=1,2$ independent of $h$, such that for an arbitrary finite interval $0 \le t \le T$,*

$$\|D_+^3 D_-^3 y\|_h \le m_1, \qquad \|D_+^2 D_-^2 dy/dt\|_h \le m_2.$$

**Proof.** In virtue of Lemma 2 and periodicity of $y_r(t)$, we get by (4)

(14) $\beta\|D_+^2 D_-^2 y\|_h \le \|d^2y/dt^2\|_h + \|D_+D_-y\|_h + \alpha\|D_+(D_-y)^{2p+1}\|_h/2p+1 \le k_3$,

where $k_3$ is a constant independent of $h$.

Now from the equality

$$D_+D_-d^2y_r/dt^2 = D_+^2 D_-(D_-y_r + \alpha(D_-y_r)^{2p+1}/2p+1) - \beta D_+^3 D_-^3 y_r,$$

Lemma 2 and (14), we obtain the following estimate

$$\|D_+^3 D_-^3 y\|_h \le m_1.$$

From the equation with respect to $v_r(t) = dy_r(t)/dt$ we get

$$\|D_+^2 D_-^2 v\|_h = \|D_+^2 D_-^2 dy/dt\|_h \le m_2.$$

using Lemma 2.

Now, in this section, it remains to show that from the solution of semi-discrete approximation (4) we may construct the desired solution in an arbitrary finite interval $0 \le t \le T$. But our method is similar to the procedure adopted by Sjöberg [2]. Then it suffices to show that we can obtain the solution by the application of Ascoli-Arzela theorem on the family of functions

$$\phi_N(t,x) = \sum_{\omega=-n}^{n} a_N(\omega,t)e^{2\pi i\omega x}, \qquad a_N(\omega,t) = (e^{2\pi i\omega x_r},\ y_N(t,x_r))_h$$

where $N = 2n+1$, $n = 1, 2, \cdots$.

By the same argument as the above one, we can prove the existence in the lower half plane $t \le 0$.

### 3. Uniqueness.

**Lemma 4.** *Let $y(t,x)$ be a solution of* (1) *with* (2) *and* (3). *Then for an arbitrary fixed strip $\{-\infty < x < \infty,\ 0 \le t \le T\}$, there exist constants $C_i$, $i=1,2,3,4$ depending only on $T, \alpha, \beta, f, g$, and their derivatives such that*

$$\|y\| \le C_1, \quad \|\partial y/\partial t\| \le C_2, \quad \max_{0\le x\le 1} |\partial y/\partial x| \le C_3, \quad \|\partial^2 y/\partial x^2\| \le C_4.$$

**Proof.** We define the energy
$E(t)=(\|\partial y/\partial t\|^2+\|\partial y/\partial x\|^2+\alpha\|(\partial y/\partial x)^{p+1}\|^2/(2p+1)(p+1)+\beta\|\partial^2 y/\partial x^2\|^2)/2$. Differentiating $E(t)$ and using periodicity of $y(t)$, we have

$$dE(t)/dt=0$$

from which it follows $\|\partial y/\partial t\|\leq C_2$, $\|\partial^2 y/\partial x^2\|\leq C_4$. Then taking into account of the inequality

$$\|y\|^2\leq 2\left(t\int_0^t\|\partial y(s)/\partial t\|^2 ds+\|f\|^2\right),$$

we obtain $\|y\|\leq C_1$. Then using Sobolev's theorem we get $\max|\partial y/\partial x|\leq C_3$. (q.e.d.)

Now we assume that $y(t,x)$ and $\hat{y}(t,x)$ are two solutions of the equation (1) satisfying the same initial conditions and (3). Then, the difference $z=y-\hat{y}$ satisfies

$$z_{tt}=z_{xx}+\alpha y_x^{2p}z_{xx}+\alpha(y_x^{2p-1}+y_x^{2p-2}\hat{y}_x+\cdots+y_x\hat{y}_x^{2p-2}+\hat{y}_x^{2p-1})\hat{y}_{xx}z_x-\beta z_{xxxx}.$$

Introducing $G(t)$ defined by

$$G(t)=(\|\partial z/\partial t\|^2+\beta\|\partial^2 z/\partial x^2\|^2+\|\partial z/\partial x\|^2)/2,$$

we get, in virtue of Lemma 4,

$$dG(t)/dt=\alpha(z_t, y_x^{2p}z_{xx})+\alpha(z_t,(y_x^{2p-1}+y_x^{2p-2}\hat{y}_x+\cdots+y_x\hat{y}_x^{2p-2}+\hat{y}_x^{2p-1})\hat{y}_{xx}z_x)$$
$$\leq \text{const. } G(t).$$

From this differential inequality and the initial conditions $z(0,x)=0$, $z_t(0,x)=0$, we can immediately conclude $z\equiv 0$ in an arbitrary fixed strip $\{-\infty<x<\infty, 0\leq t\leq T\}$.

This completes the proof of the theorem.

Up to now we have not succeeded in proving the global existence for the following equation:

$$\partial^2 y/\partial t^2=(1+\alpha(\partial y/\partial x)^{2p+1})\partial^2 y/\partial x^2-\beta\partial^4 y/\partial x^4,$$

where $\alpha$ and $\beta$ are positive constants and $p=0,1,2,\cdots$.

## References


[1] Zabusky, N. J.: A synergetic approach to problems of nonlinear dispersive wave propagation and interaction. Nonlinear Partial Differential Equations, W. Ames, ed., Academic Press, New York, pp. 223–258 (1967).

[2] Sjöberg, A.: On the Korteweg - de Vries equation. Uppsala Univ. Dept. of Computer Sci., Report (1967).

[3] Nisida, T.: On some semilinear dispersive equation (to appear).