99. *Propagation of Chaos for Certain Markov Processes of Jump Type with Nonlinear Generators.* I

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1. Introduction. Let Q be a set endowed with a σ-field $\mathcal{F}$ of its subsets such that each single point set $\{x\}$ is in $\mathcal{F}$, and denote by $\mathcal{P}$ the set of probability measures on $(Q, \mathcal{F})$. Suppose we are given a kernel $A_f(x, \Gamma)$ indexed by $f \in \mathcal{P}$ with the form:

$$A_f(x, \Gamma)=\sum_{n=1}^{\infty}\int\cdots\int A_n^{(x_1,\cdots,x_n)}(x, \Gamma)f(dx_1)\cdots f(dx_n), \quad \Gamma \in \mathcal{F},$$

and assume that the following 3 conditions are satisfied.

(i) For each $n\geq 1$, $x, x_1, \cdots, x_n \in Q$, $A_n^{(x_1,\cdots,x_n)}(x, \cdot)$ is a bounded signed measure on $(Q, \mathcal{F})$ which is nonnegative outside $\{x\}$ and has zero total mass.

(ii) For each $n\geqslant 1$, $\Gamma \in \mathcal{F}$, $A_n^{(x_1,\cdots,x_n)}(x, \Gamma)$ is a measurable function of $(x, x_1, \cdots, x_n)$, symmetric in $(x_1, \cdots, x_n)$ when x is fixed, and $A_n^{(x_1,\cdots,x_n)}(x, \{x\})$ is measurable in $(x, x_1, \cdots, x_n)$.

(iii) $q=\sum_{n=1}^{\infty} q_n<\infty$, where $q_n=\sup_{x,x_1,\cdots,x_n\in Q} A_n^{(x_1,\cdots,x_n)}(x, Q-\{x\})$.

We are concerned with the following nonlinear equation:

$$\frac{du(t)}{dt}=Au(t), \qquad u(0+)=f, \tag{1.1}$$

where the initial value f and the solution u, for each $t>0$, are in $\mathcal{P}$, and $(Au)(\cdot)=\int_Q A_u(x, \cdot)u(dx)$.

Denote by Q^n ($\mathcal{F}^n$) the n-fold product space $Q\times\cdots\times Q$ (σ-field $\mathcal{F}\times\cdots\times\mathcal{F}$) of Q ($\mathcal{F}$), and let A_n be a linear operator from the space $\mathfrak{M}_n$ of bounded signed measures on $(Q^n, \mathcal{F}^n)$ into itself defined by

$$(A_n u)(\Gamma)=\int_{Q^n} u(dx_1\cdots dx_n)\sum_{N=1}^{n-1} n^{-N}\sum_{i,i_1,\cdots,i_N}^{(n)}\int_Q A_N^{(x_{i_1},\cdots x_{i_N})}(x_i, \Gamma),$$

where $\sum_{i,i_1,\cdots,i_N}^{(n)}$ is the sum with respect to all $(i, i_1, \cdots, i_N)$ such that $i, i_1, \cdots, i_N$ are all different and $1\leq i, i_1, \cdots, i_N\leq n$; χ_Γ is the indicator function of $\Gamma \in \mathcal{F}^n$, and the notation $A_N^{(x_{i_1},\cdots,x_{i_N})}(x_i, \varphi)$ for $\varphi=\varphi(x_1, \cdots \cdots, x_n)$ stands for

$$\int_Q A_N^{(x_{i_1},\cdots,x_{i_N})}(x_i, dx)\varphi(\cdots, x_{i-1}, x, x_{i+1}, \cdots).$$

Consider the linear equation for $n=2, 3, \cdots$:

$$(1.2)\qquad \frac{du_n(t)}{dt}=A_nu_n(t),\quad u_n(0+)=f^n,$$

where f^n denotes the n-fold outer product $f\otimes\cdots\otimes f$ of the same f in (1.1). The propagation of chaos, first discovered by Kac [1] for the 1-dimensional Maxwellian gas, asserts that the solution $u_n(t)$ of (1.2) tends as $n\uparrow\infty$ to the infinite outer product $u(t)^\infty=u(t)\otimes u(t)\otimes\cdots$ of the solution $u(t)$ of (1.1). The proof of this is our object. In this direction, H. P. McKean [2] [3] first pointed out that Kac's propagation of chaos should hold for a wide class of Markov processes with nonlinear generators. As for Markov processes of pure jump type with nonlinear generators, D. P. Johnson [4] studied the 2-state case ($Q=\{\pm 1\}$), and T. Ueno [5] treated the general state case. In both cases the following condition (A_0) is assumed:

(A_0) there exists a constant L such that $\sum_{n=1}^{\infty} n^p q_n \leq p\,!\,L^p,\ p=1,2,\cdots.$

In this and the subsequent papers, the condition (A_0) will be replaced by the following weaker condition (A):

$$\text{(A)}\qquad \int_{1-\varepsilon}^{1}\frac{dt}{q-\sum_{n=1}^{\infty}q_nt^n}=\infty\qquad \text{for}\quad \varepsilon\in(0,1),$$

and the propagation of chaos will be proved as a convergence theorem for certain linear semigroups. In the arguments, a simple branching model is involved, and the condition (A) means that this branching process does not blow up. Full proofs will be published elsewhere.

2. Linear semigroups associated with the equation (1.1). We introduce several notations. Let Φ^n be the Banach space of bounded $\mathcal{F}^n$-measurable real valued functions on Q^n with the supremum norm $\|\cdot\|$, and regarding $\varphi\in\Phi^n$ as a function on the infinite product space $Q^\infty=Q\times Q\times\cdots$, set $\Phi^\infty=\bigcup_{n=1}^{\infty}\Phi^n$ and let Φ be the completion of Φ^∞ with respect to the supremum norm. We set, for each $p=0,1,\cdots$,

$\mathfrak{N}_p=$ the closure of

$$\left\{\begin{matrix}\varphi\in\Phi^\infty:\ \text{for each } (x_1,\cdots,x_p)\in Q^p,\ \text{and } f\in\mathcal{P}\\ \iint\cdots\varphi(x_1,\cdots,x_p,x_{p+1},x_{p+2},\cdots)f(dx_{p+1})f(dx_{p+2})\cdots=0\end{matrix}\right\},$$

and $\hat{\Phi}_p=\Phi/\mathfrak{N}_p$; this is a Banach space with the usual induced norm $\|\hat{\varphi}\|_p=\inf_{\varphi\in\hat{\varphi}}\|\varphi\|$. For $0\leq p$, $p_1\leq p_2$, denote by θ_p ($\theta_{p_1p_2}$) the natural mapping from Φ ($\hat{\Phi}_{p_2}$) onto $\hat{\Phi}_p$ ($\hat{\Phi}_{p_1}$), and set $\hat{\Phi}_p^n=\theta_p(\Phi^n)$, $\hat{\Phi}_p^\infty=\theta_p(\Phi^\infty)$. The integral of a function φ with respect to a measure f is denoted by $\langle f,\varphi\rangle$. Also, for $\hat{\varphi}\in\hat{\Phi}_p$, $f\in\mathcal{P}$, we set $\langle f^\infty,\hat{\varphi}\rangle_p(x_1,\cdots,x_p)$ $=\iint\cdots\varphi(x_1,\cdots,x_p,x_{p+1},x_{p+2},\cdots)f(dx_{p+1})f(dx_{p+2})\cdots$, where $\varphi\in\hat{\varphi}$; when $p=0$, we write simply $\langle f^\infty,\hat{\varphi}\rangle$ for the above. Finally we introduce a multiplication $\hat{\varphi}\otimes\hat{\psi}(\in\hat{\Phi}_{p+p'})$ for $\hat{\varphi}\in\hat{\Phi}_p$ and $\hat{\psi}\in\hat{\Phi}_{p'}$ as follows. If $\varphi\in\hat{\varphi}$, $\psi\in\hat{\psi}$, we set

$$\hat{\varphi}\otimes\hat{\psi}=\theta_{p+p'}\{\varphi(x_1,\cdots,x_p,x_{p+p'+1},x_{p+p'+3},\cdots)\psi(x_{p+1},\cdots,x_{p+p'},x_{p+p'+2},x_{p+p'+4},\cdots)\}.$$

Then, this definition makes sense, since the right hand side does not depend upon a particular choice of the representatives φ, ψ of $\hat{\varphi}$ and $\hat{\psi}$.

We introduce a linear operator $D_p:\hat{\Phi}_p^\infty\to\hat{\Phi}_p$ for each $p=0,1,\cdots$, by

$$D_p\hat{\varphi}=\theta_p\left\{\sum_{N=1}^\infty\sum_{i=1}^m A_N^{(x_{m+p+1},\cdots,x_{m+p+N})}(x_i,\varphi)\right\},\quad \hat{\varphi}=\theta_p\varphi,\quad \varphi\in\Phi^m.$$

It can be proved that the right hand side depends only on $\hat{\varphi}\in\hat{\Phi}_p$, but not on a particular choice of $\varphi\in\hat{\varphi}$. Now, our task in this section is to construct a semigroup $\{H_p^t\}$ on $\hat{\Phi}_p$ with generator D_p (or more precisely, certain closed extension of D_p) under the condition (A). Since D_p is unbounded, we need some tricks for this. Denote by $\mathbf{Q}$ the direct sum $Q^1+Q^2+\cdots$. The restriction onto Q^n of a function $\boldsymbol{\varphi}$ defined on $\mathbf{Q}$ is denoted by φ_n or $(\boldsymbol{\varphi})_n$, and we set $\mathbf{\Phi}=\{\boldsymbol{\varphi}:\varphi_n\in\Phi^n$ for each $n\geqslant 1\}$. $\mathbf{\Phi}$ is a Fréchet space with seminorms $\|\boldsymbol{\varphi}\|_n=\max_{1\le k\le n}\|\varphi_k\|$, $n\geqslant 1$. We next rewrite the kernel $A_n^{(x_1,\cdots,x_n)}(x,\Gamma)$:

$$A_n^{(x_1,\cdots,x_n)}(x,\Gamma)=q_n\{\Pi_n(x,x_1,\cdots,x_n,\Gamma)-\delta(x,\Gamma)\}$$

where

$$\Pi_n(x,x_1,\cdots,x_n,\Gamma)=\begin{cases} q_n^{-1}A_n^{(x_1,\cdots,x_n)}(x,\Gamma), & x\notin\Gamma \\ q_n^{-1}\{q_n+A_n^{(x_1,\cdots,x_n)}(x,\{x\})\}, & \Gamma=\{x\},\end{cases}$$

and define a linear operator $\mathbf{D}_p:\mathbf{\Phi}\to\mathbf{\Phi}$ by

$$\begin{aligned}(\mathbf{D}_p\boldsymbol{\varphi})_k(x_1,\cdots,x_k)&=\sum_{m=1}^{k-p-1}\sum_{i=1}^m q_{k-p-m}\Pi_{k-p-m}(x_i,x_{p+m+1},\cdots,x_k,\varphi_m)-kq\varphi_k \\ &\qquad\qquad \text{for } k>p+1 \\ &=-kq\varphi_k \qquad\qquad \text{for } 1\le k\le p+1.\end{aligned}$$

Let $\{\mathbf{H}_p^t\}$ be the semigroup on $\mathbf{\Phi}$ with generator $\mathbf{D}_p$; this is obtained by the usual exponential sum. For $\boldsymbol{\varphi}\in\mathbf{\Phi}$ such that $\sum_{n=1}^\infty\|\varphi_n\|<\infty$, we define $\gamma_p\boldsymbol{\varphi}\in\hat{\Phi}_p$ by $\gamma_p\boldsymbol{\varphi}=\theta_p\left\{\sum_{n=1}^\infty\varphi_n(x_1,\cdots,x_n)\right\}$. The key point in our argument is the following relation between $\mathbf{D}_p$ and D_p:

$$\gamma_p\mathbf{D}_p\boldsymbol{\varphi}=D_p\gamma_p\boldsymbol{\varphi}$$

for $\boldsymbol{\varphi}\in\mathbf{\Phi}$ such that $\varphi_n=0$ except for finitely many n, and this implies the *formal* formula:

$$\gamma_p\mathbf{H}_p^t\boldsymbol{\varphi}=\gamma_p\sum_{n=0}^\infty\frac{t^n}{n!}\mathbf{D}_p^n\boldsymbol{\varphi}=\sum_{n=0}^\infty\frac{t^n}{n!}D_p^n\gamma_p\boldsymbol{\varphi}=H_p^t\gamma_p\boldsymbol{\varphi}. \tag{2.1}$$

Under the assumption (A) this formal argument can actually be made rigorous, and we can define a linear operator H_p^t on $\hat{\Phi}_p$ by the left hand side of (2.1), which has nice properties as will be summarized in the following theorem.

Theorem 1. *Under the assumption* (A), *for each* $p=0, 1, \cdots$, $\{H_p^t\}$ *is a strongly continuous contraction semigroup on* $\hat{\Phi}_p$ *and has the following properties.*

i) $H_p^t \mathbf{1} = \mathbf{1}$

ii) $\lim_{t \downarrow 0} t^{-1}(H_p^t \hat{\varphi} - \hat{\varphi}) = D_p \hat{\varphi} \quad \text{for } \hat{\varphi} \in \hat{\Phi}_p^\infty$

iii) $H_{p+p'}^t(\hat{\varphi} \otimes \hat{\psi}) = (H_p^t \hat{\varphi}) \otimes (H_{p'}^t \hat{\psi}) \quad \text{for } \hat{\varphi} \in \hat{\Phi}_p,\ \hat{\psi} \in \hat{\Phi}_{p'}$,

and in particular, $\{H_0^t\}$ *is a multiplicative semigroup on the Banach algebra* $\hat{\Phi}_0$.

iv) $H_p^t \theta_{pp'} = \theta_{pp'} H_{p'}^t \quad \text{if } 0 \leq p \leq p'$.

v) *For each* $f \in \mathcal{P}$, *the formula*

$$\langle f^\infty, H_0^t \hat{\varphi} \rangle = \langle u(t), \varphi \rangle, \quad \hat{\varphi} = \theta_0 \varphi, \quad \varphi \in \Phi^1$$

defines a probability measure $u(t)$ *which is the unique solution of* (1.1).

vi) *For each* $f \in \mathcal{P}$, *the formula*

$$\langle f^\infty, H_1^t \hat{\varphi} \rangle_1(x) = \langle P_f(t, x, \cdot), \varphi \rangle, \quad \hat{\varphi} = \theta_1 \varphi, \quad \varphi \in \Phi^1$$

defines a probability measure $P_f(t, x, \cdot)$; $\{P_f(t, x, \cdot)\}$ *satisfies the Kolmogorov-Chapman equation*:

$$P_f(t+s, x, \Gamma) = \int_Q P_f(t, x, dy) P_{u(t)}(s, y, \Gamma)$$

and $u(t, \Gamma) = \int_Q P_f(t, x, \Gamma) f(dx)$ *is the solution of* (1.1).

Remark. $\{H_p^t\}$ for $p \geqslant 2$ does not have much sense compared with $\{H_0^t\}$ and $\{H_1^t\}$. Also, $\{H_1^t\}$ determines $\{H_p^t\}$ for all $p \geqslant 0$ by iii) and iv).

References

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