97. A Remark on the Π-imbedding of Homotopy Spheres

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Let Θ_n be the group of homotopy n-spheres and $\tilde{S}^n$ be an element of Θ_n. $\tilde{S}^n$ represents an element of a subgroup $\Theta_n(\partial\pi)$ of Θ_n if and only if $\tilde{S}^n$ is the boundary of a parallelizable manifold.

It is known that every $\tilde{S}^{13}$ is imbeddable in the 17-dimensional unit sphere S^{17} with a trivial normal bundle (Katase [3]). (Such an imbedding is called a π-imbedding.) But in the case of codimension 3, it has been unknown whether the π-imbedding exists or not. The result of this paper is that there exists a 13-dimensional homotopy sphere $\tilde{S}^{13}$ which is not π-imbeddable in S^{16}.

1. Suppose that $\tilde{S}^n$ is π-imbedded in S^{n+k} $(3\leqq k<n)$. Then the tubular neighbourhood of $\tilde{S}^n$ in S^{n+k} and its boundary is easily seen to be diffeomorphic to $S^n\times D^k$ and $S^n\times S^{k-1}$ respectively (here D^k is the closed unit disk in euclidean k-space and is bounded by S^{k-1}). Moreover, $\tilde{S}^n$ is isotopic to an $\tilde{S}^n_1$ which lies in $S^n\times S^{k-1}\subset S^{n+k}$ with normal $(k-1)$-frame $\mathcal{F}$ in $S^n\times S^{k-1}$ and is homotopic, in $S^n\times S^{k-1}$, to $S^n\times x_0$ for some $x_0\in S^{k-1}$ (Levine [6]). The Pontrjagin-Thom construction with respect to a normal $(k-1)$-frame $\mathcal{F}$ on $\tilde{S}^n_1$ in $S^n\times S^{k-1}$ yields a map

$$\varphi\,;\ S^n\times S^{k-1}\longrightarrow S^{k-1}$$

which maps $\tilde{S}^n_1$ to a point p in S^{k-1} (see, for example, Kervaire [4]).

Suppose that φ can be extended to a map

$$\Phi'\,;\ S^{n+k}-\mathrm{Int}\, S^n\times D^k\longrightarrow S^{k-1}.$$

Then we can approximate it by a smooth map Φ keeping φ fixed.

Since we may consider p as a regular value of Φ, $\Phi^{-1}(p)$ or at least the component of $\tilde{S}^n_1$ in $\Phi^{-1}(p)$ is an $(n+1)$-dimensional submanifold of S^{n+k} with a trivial normal bundle and its boundary is $\tilde{S}^n_1$. Therefore $\tilde{S}^n$ bounds a parallelizable manifold, i.e., $\tilde{S}^n$ is an element of $\Theta_n(\partial\pi)$.

2. Now we consider the obstructions to extending φ over $S^{n+k}-\mathrm{Int}(S^n\times D^k)$ which lie in the cohomology groups

$$H^r(S^{n+k}-\mathrm{Int}(S^n\times D^k),\ S^n\times S^{k-1}\,;\ \pi_{r-1}(S^{k-1})).$$

Lemma. *The obstructions to such an extension are zero for* $r \neq n+k$.

Proof. Consider the cohomology exact sequence of the pair $(S^{n+k} - \mathrm{Int}(S^n \times D^k),\ S^n \times S^{k-1})$. Since the inclusion map

$$\iota;\ y_0 \times S^{k-1} \longrightarrow S^{n+k} - \mathrm{Int}(S^n \times D^k), \quad \text{for some } y_0 \in S^n,$$

is a homotopy equivalence, we see that $H^r(S^{n+k} - \mathrm{Int}(S^n \times D^k),\ S^n \times S^{k-1})$ are zero except for $r = n+1$ and $n+k$. As for the case of $r = n+1$, consider the following commutative diagram:

$$\begin{array}{ccccc} \pi_{n+1}(S^{n+k} - \mathrm{Int}(S^n \times D^k), S^n \times S^{k-1}) & \rightarrow & \pi_n(S^n \times S^{k-1}) & \xrightarrow[i_*]{} & \pi_n(S^{n+k} - \mathrm{Int}(S^n \times D^k)) \\ H \downarrow \cong & & H \downarrow & & \\ H_{n+1}(S^{n+k} - \mathrm{Int}(S^n \times D^k), S^n \times S^{k-1}) & \xrightarrow{\cong} & H_n(S^n \times S^{k-1}) & & \end{array}$$

where H is the Hurewicz homomorphism.

Since $i_* = \iota_* \circ (p_2)_*$, where $p_2 \colon S^n \times S^{k-1} \rightarrow y_0 \times S^{k-1}$ is the projection on the second factor, the boundary of the generating cycle of $H_{n+1}(S^{n+k} - \mathrm{Int}(S^n \times D^k),\ S^n \times S^{k-1})$ is homologous and homotopic to $\tilde{S}^n_1$ in $S^{n+k} - \mathrm{Int}(S^n \times D^k)$ and φ maps $\tilde{S}^n_1$ to a point p in S^{k-1}. Hence φ can be extended over $(S^{n+k} - \mathrm{Int}(S^n \times D^k))^{(n+1)} \cup S^n \times S^{k-1}$ and the obstruction appears only in the dimension $n+k$.

Applying this lemma, we obtain

Theorem. *There exists a* 13*-dimensional homotopy sphere* $\tilde{S}^{13}$ *which is not* π*-imbeddable in* S^{16}.

Proof. Suppose that the generator $\tilde{S}^{13}$ of $\Theta_{13} \cong Z_3$ is π-imbeddable in S^{16}. Since $\tilde{S}^{13}$ does not bound a parallelizable manifold, the obstruction σ to extending φ over $S^{16} - \mathrm{Int}(S^{13} \times D^3)$ is a non-zero element of $H^{16}(S^{16} - \mathrm{Int}(S^{13} \times D^3),\ S^{13} \times S^2;\ \pi_{15}(S^2)) \cong \pi_{15}(S^2) \cong Z_2 + Z_2$ (Toda [7]). The obstruction over the connected sum of pairs $(S^{16}, S^{13} \times D^3) \# (S^{16}, S^{13} \times D^3)$ (see, for example, Haefliger [1] where the disk pair (D^{16}, D^{13}) must be imbedded so that we may obtain $\tilde{S}^{13}_1 \# \tilde{S}^{13}_1$) is twice of σ and $2\sigma = 0$. This contradicts the fact that $\tilde{S}^{13} \# \tilde{S}^{13}$ is not an element of $\Theta_{13}(\partial\pi) = 0$. Therefore $\tilde{S}^{13}$ is not π-imbeddable in S^{16}.

Addendum to the preceeding paper [3].

Let $\tilde{S}^n$ ($\in \Theta_n$) correspond (modulo J-image) to an element α of $\pi_{N+n}(S^N)$ for sufficiently large N and let $\tilde{S}^n$ be π-imbedded in S^{n+k}, then α is an $(N-k)$-fold suspension element (modulo J-image). Applying this fact, we see that there exist homotopy 10-, 14-, 17- and 18-spheres which are not π-imbeddable in S^{15}, S^{21}, S^{28} and S^{29} respectively.

On the other hand, following the method of Hsiang, Levine and

Szczarba [2], we obtain that every homotopy 17- and 18-sphere is π-imbeddable in S^{29} and S^{30} respectively.

(Note that Theorem (1.2) in [2] can also be proved for $n=18$.)

Thus we rewrite the table in [3].

Table

n	8	9	10	13	14	15	16	17	18
order of Θ_n	2	8	6	3	2	16256	2	16	16
order of $\Theta_n(\partial\pi)$	1	2	1	1	1	8128	1	2	1
k	4	4	6	4	7~8	3~4	14	12	12

(k is the smallest codimension with which every homotopy n-sphere is π-imbeddable.)

References

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