95. *A Class of Purely Discontinuous Markov Processes with Interactions. II*[1)]

By Tadashi UENO
Department of Mathematics, Faculty of General Education,
University of Tokyo

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1. Consider a branching model as follows. Let b_0 be a trivial branch or a pole, and let (b_0, b_0) be as in Fig. 1. T is the set of all branches which grow downward with binary branching points.

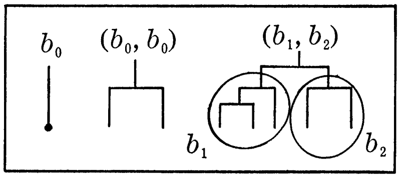

Fig. 1

$b=(b_1, b_2)$ is the branch which have b_1 and b_2 on the left and the right hand of the highest branching point, respectively. Length $l(b)$ and a number $\sharp(b)$ are defined by

$$l(b_0)=0, \quad l(b)=1+\max(l(b_1), l(b_2)),$$
$$\sharp(b_0)=1, \quad \sharp(b)=\sharp(b_1)+\sharp(b_2), \quad \text{for } b=(b_1, b_2).$$

T_n denotes the set of all branches with length at most n. $b_0(x)$ is the trivial branch with variable $x \in R$ at the bottom, and $b(x)$ is the branch b with variable x at the bottom of the left extreme point. The correspondence $b \rightleftarrows b(x)$ is clearly one to one. $T(x)$, $l(b(x))$, $\sharp(b(x))$, $T_n(x)$ $(b_1(x), b_2)$, for $b_1(x) \in T(x)$ and $b_2 \in T$, are defined similarly. The following is clear by induction:

$$T_0=\{b_0\}, \quad T_{n+1}=\{b_0\} \cup \{(b_1, b_2), b_1, b_2 \in T_n\}$$
$$(1) \qquad T_0(x)=\{b_0(x)\}, \quad T_{n+1}(x)=\{b_0(x)\} \cup \{(b_1(x), b_2), b_1(x) \in T_n(x), b_2 \in T_n\}$$
$$T=\bigcup_{n=0}^{\infty} T_n, \quad T(x)=\bigcup_{n=0}^{\infty} T_n(x).$$

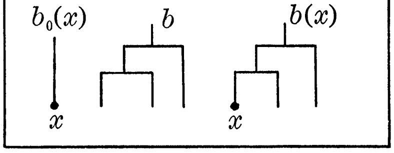

Fig. 2

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Recall the scheme in [3] and assume $q(x_1|t,x)\equiv q(t,x)$, to have $\pi^0(x_1|t,x,E)\equiv\pi(x_1|t,x,E)$ for an intuitive explanation. Define

$$P^{(f)}(s,b_0,t,E)=\int_R f(dx)e^{-\int_s^t q(\sigma,x)d\sigma}\delta_x(E),$$

$$P^{(f)}(s,b_0(x),t,E)=e^{-\int_s^t q(\sigma,x)d\sigma}\delta_x(E)$$

$$(2)\qquad P^{(f)}(s,b,t,E)=\int_s^t dr\int_{R^2} P^{(f)}(s,b_1,r,dy)P^{(f)}(s,b_2,r,dx_1)q(r,y)\times\int_E \pi(x_1|r,y,dz)e^{-\int_r^t q(\sigma,z)d\sigma}$$

$$P^{(f)}(s,b(x),t,E)=\int_s^t dr\int_{R^2} P^{(f)}(s,b_1(x),r,dy)P^{(f)}(s,b_2,r,dx_1)q(r,y)\times\int_E \pi(x_1|r,y,dz)e^{-\int_r^t q(\sigma,z)d\sigma},$$

for $b=(b_1,b_2)$ and $b(x)=(b_1(x),b_2)$. We have, by induction,

$$(3)\qquad P^{(f)}(s,b,t,E)=\int_R f(dx)P^{(f)}(s,b(x),t,E).$$

Then, (1), (2), (3) and the definition of $S_n^{(f)}(s,x,t,E)$ in [3] easily imply

Theorem 1. *For each substochastic f,*

$$(4)\qquad \begin{aligned} S_n^{(f)}(s,x,t,E)&=\sum_{b(x)\in T_n(x)} P^{(f)}(s,b(x),t,E),\\ S_n^{(f)}(s,t,E)&=\sum_{b\in T_n} P^{(f)}(s,b,t,E).\end{aligned}$$

$$(5)\qquad \begin{aligned} P^{(f)}(s,x,t,E)&=\sum_{b(x)\in T(x)} P^{(f)}(s,b(x),t,E),\\ P_{s,t}^{(f)}(E)&=\sum_{b\in T} P^{(f)}(s,b,t,E).\end{aligned}$$

2. Intuitive meanings of the quantities are as follows. $P^{(f)}(s,b_0(x),t,E)$ is the probability that the particle at point x stands still in the set E from time s to t. $P^{(f)}(s,b(x),t,E)$ is the probability that the particle under observation, started at point x at time s, is in the set E at time t after the interactions with other $\#(b(x))-1$ similar particles, which started at time s with the initial distribution f independently. Here, the branch $b(x)$ determines the order of interactions, and $q(t,x)$ and $\pi(x_1|t,x,E)$ determine the waiting times until interactions and the hitting measures, respectively. $P^{(f)}(s,b,t,E)$ is the probability of the same event except that the particle under observation also starts with initial distribution f. Thus, $S_n^{(f)}(s,x,t,E)$ is the probability that the particle, starting at x at time s, is found in E at time t after the interactions determined by the branches with length at most n. The minimal solution $P^{(f)}(s,x,t,E)$ is the sum of all these possibilities.

In case $q(x_1|t,x)\not\equiv q(t,x)$, or when there is no function like $q(t,x)$, the situation is not so simple. Even the probability that one particle stands still at point x from time s to t reflects the infinitely many jumps of other particles. In fact, when the equation (3) of [3] has a stochastic solution, this probability is determined by $\int_R P_{s,t}^{(f)}(dx_1)q(x_1|t, x)$, where the effect of all jumps is implicitly included in $P_{s,t}^{(f)}(\cdot)$. The independence assumption on $q(x_1|t, x)$ from x_1 cancel the effect of interactions on the probability of standing still.

The modification (2′) of [3], in case $q(x_1|t, x)\not\equiv q(t, x)$, is technically based on the fact that $\pi(x_1|t, x, \{x\})\equiv 0$ is not necessary for our method. Intuitively, this amounts to let the particles jump at the time determined by $q(t, x)$, earlier than the proper time governed by $q(x_1|t, x)$, while the particle jumps back instantly to the starting point x with probability $1-q(t, x)^{-1}q(x_1|t, x)$ and jumps into $R-\{x\}$ with reduced hitting measure $q(t, x)^{-1}q(x_1|t, x)\pi^0(x_1|t, x, E)$.

A characteristic of our model is that the interactions take place always between new particles.[2] This can be explained in terms of the original gas model of Boltzmann, where the gas is so dilute that the second or later interactions between the same particles can be ignored.[3]

3. A version of the *Kolmogorov backward equation* holds.

Theorem 2. *For a fixed time* s_0 *and a substochastic measure* f,

$$(6)\qquad \frac{d}{ds}P^{(f_s)}(s, x, t, E)=-q(s, x)\int_R\left(\int_R P_{s_0,s}^{(f)}(dx_1)\pi(x_1|s, x, dy)-\delta_x(dy)\right)P^{(f)}(s, y, t, E),^{4)}$$

where $f_s(\cdot)=P_{s_0,s}^{(f)}(\cdot)=\int_R f(dx)P^{(f)}(s_0, x, s, \cdot)$, *and* $s_0\leq s<t$.

Outline of the proof. Replace f by f_s in (7) of [3] and write

$$\pi(s, x, E)\equiv\int_R P_{s_0,s}^{(f)}(dx_1)\pi(x_1|s, x, E)\equiv\int_R f_s(dx_1)\pi(x_1|s, x, E)$$

and $P(s, x, t, E)\equiv P^{(f_s)}(s, x, t, E)$. Then, (7) of [3] implies that $P(s, x, t, E)$ is a solution of

$$(7)\qquad P(s,x,t,E)-\delta_x(E)=\int_s^t dr\int_R P(s,x,r,dy)q(r,y)(\pi(r,y,E)-\delta_y(E)),$$

for *bounded* E. Moreover, $P(s, x, t, E)$ is the *minimal solution*. In fact, the minimal solution $P^m(s, x, t, E)$ of (7) is approximated from below by $S_n(s, x, t, E)$: $S_0(s, x, t, E)=e^{-\int_s^t q(\sigma,x)d\sigma}\delta_x(E)$,

$$S_{n+1}(s, x, t, E)=e^{-\int_s^t q(\sigma,x)d\sigma}\delta_x(E)+\int_s^t dr\int_R S_n(s, x, r, dy)q(r, y)\times\int_E \pi(r, y, dz)e^{-\int_r^t q(\sigma,z)d\sigma}.$$

But, since we have for all n,

2) Compare with the definition of D or e^{tD} in [2], where new variables are induced by each application of D or e^{tD}. This reflects the situation here.

3) This explanation owes to McKean.

4) The differentiation in s with fixed f is clearly meaningless, intuitively.

$$\pi(r, x, E)=\int_R f_r(dx_1)\pi(x_1 | r, x, E)$$
$$=\int_R P^{(f)}(s, r, dx_1)\pi(x_1 | r, x, E)\geq\int_R S_n^{(fs)}(s, r, dx_1)\pi(x_1 | r, x, E),$$

$S_{n+1}(s, x, t, E)\geq S_{n+1}^{(fs)}(s, x, t, E)$ by induction, in view of (5) in [3]. This implies $P^m(s, x, t, E)\geq P^{(fs)}(s, x, t, E)\equiv P(s, x, t, E)$, and hence the equality by the minimal property of $P^m(s, x, t, E)$. Then, take for granted that the Feller's results in [1] hold true for a model with sub-stochastic hitting measure $\pi(r, x, E)$.[5] Then, the minimal solution of (17) automatically satisfies the backward equation

$$\frac{d}{ds}P(s, x, t, E)=-q(s, x)\int_R (\pi(s, x, dy)-\delta_x(dy))P(s, x, t, E),$$

which is exactly (6) by changing the notation.

4. The results through **1-3** hold true for the general model in 4 of [3] with clear modifications. Moreover, the contents of this paper and [3] can be extended to a model determined by

$$P^{(f)}(s, x, t, E)=P_0(s, x, t, E)$$
$$(8)\qquad +\int_s^t dr\int_{R^2} P^{(f)}(s, x, r, dy)P_{s,r}^{(f)}(dx_1)q(r, y)$$
$$\times\int_R \pi(x_1 | r, y, dz)P_0(r, z, t, E),$$

where $P_0(s, x, t, E)$ is the transition probability induced by the killing by $q(t, x)$ from a transition probability $P(s, x, t, E)$ of a Markov process moving on R. Here, we restricted the model to the binary interacting case for simplicity. The backward equation for this model is based on an extension of Feller [1].

These results will be discussed later.

References

[1] W. Feller: On the integro-differential equations of purely discontinuous Markov processes. Trans. Amer. Math. Soc., **48**, 488–515 (1940). [erratum, **58**, p. 474]

[2] T. Ueno: A class of Markov processes with bounded, non-linear generators (to appear in Japanese J. Math.).

[3] ——: A class of purely discontinuous Markov processes with interactions. I. Proc. Japan Acad., **45**, 348–353 (1969).

5) The gap will be completed as a special case of a subsequent work related with (8) in 4.