# 94. On Dirichlet Spaces and Dirichlet Rings

By Masatoshi FUKUSHIMA
Department of the Foundations of Mathematical Sciences,
Tokyo University of Education

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In [2], we have introduced the notion of the Dirichlet space relative to an $L^2$-space (we will call this an $L^2$-Dirichlet space). The purpose of this paper is to derive a normed ring (called a Dirichlet ring) from any given $L^2$-Dirichlet space in the similar manner as Royden ring [5] from the space of functions with finite Dirichlet integrals. Dirichlet rings will enable us to define a natural equivalence relation among the collection of all $L^2$-Dirichlet spaces. We will discuss elsewhere the problem to find out nice versions from each equivalence class ([3]).

## § 1. $L^2$-Dirichlet spaces and $L^2$-resolvents.

We call $(X, m, \mathcal{F}, \mathcal{E})$ a complex *$L^2$-Dirichlet space* (in short, a *D-space*) if the following conditions are satisfied.

(1.1) $X$ is a locally compact Hausdorff space.

(1.2) $m$ is a Radon measure on $X$.

(1.3) $\mathcal{F}$ is a linear subspace of complex $L^2(X)=L^2(X; m)$, two functions being identified if they coincide $m$-a.e. on $X$. $\mathcal{E}$ is a non-negative definite bilinear form on $\mathcal{F}$ and, for each $\alpha>0$, $\mathcal{F}$ is a complex Hilbert space with inner product

$$\mathcal{E}^{\alpha}(u, v)=\mathcal{E}(u, v)+\alpha(u, v)_X,$$

where $(u, v)_X$ is the inner product in $L^2(X)$-sense.

(1.4) Each normal contraction operates on $(\mathcal{F}, \mathcal{E})$: if $u \in \mathcal{F}$ and a measurable function $v$ satisfies

$$|v(x)| \leqq |u(x)|, \quad |v(x)-v(y)| \leqq |u(x)-u(y)| \ m\text{-a.e},$$

then $v \in \mathcal{F}$ and $\mathcal{E}(v, v) \leqq \mathcal{E}(u, u)$.

Let $(X, m)$ be as above. We call a family of linear bounded symmetric operators $\{G_\alpha, \alpha>0\}$ on $L^2(X)$ an *$L^2$-resolvent* iff it satisfies the resolvent equation and it is sub-Markov: for each $\alpha>0$, $G_\alpha$ translates each real function into a real function and $0 \leqq \alpha G_\alpha u \leqq 1$ $m$-a.e for $u \in L^2(X)$ such that $0 \leqq u \leqq 1$ $m$-a.e.

There is a one-to-one correspondence between the class of $D$-spaces and the class of $L^2$-resolvents ([2]).

In fact, with any $D$-space $(X, m, \mathcal{F}, \mathcal{E})$, we can associate an $L^2$-resolvent by the equation

(1.5) $$\mathcal{E}^{\alpha}(G_\alpha u, v)=(u, v)_X \quad \text{for any} \quad v \in \mathcal{F},$$

where $u$ is any element of $L^2(X)$.

Conversely, for any $L^2(X; m)$-resolvent $\{G_\alpha, \alpha>0\}$, a $D$-space can be defined by

$$\mathcal{E}(u,u)=\lim_{\beta\to+\infty}\beta(u-\beta G_\beta u, u)_X,\ u\in L^2(X), \tag{1.6}$$

$$\mathcal{F}=\{u\in L^2(X);\ \mathcal{E}(u,u)<+\infty\}. \tag{1.7}$$

We note that each normal contraction operates on this space because of the following lemma.

**Lemma 1.** *For $u\in L^2(X)$, $\alpha\geqq 0$, $\beta>0$,*

$$\beta(u-\beta G_{\beta+\alpha}u, u)_X=\frac{1}{2}\beta^2\int_X\int_X\sigma_{\beta+\alpha}(dx,dy)\,|u(x)-u(y)|^2$$
$$+\frac{\beta}{\beta+\alpha}\alpha(u,u)_X+\frac{\beta}{\beta+\alpha}\beta\int_X(1-(\beta+\alpha)k_{\beta+\alpha}(x))\,|u(x)|^2m(dx).$$

*Here, $\sigma_\beta$ is a Radon measure satisfying*

$$(u, G_\beta v)_X=\int_X\int_X\sigma_\beta(dx,dy)u(x)\overline{v(y)},\qquad u,v\in L^2(X),$$

*and $k_\beta$ is a Radon-Nikodym derivative of the measure $\sigma_\beta(\cdot\times X)$ with respect to $m(\cdot)$.*

The correspondence defined by (1.5) and that defined by (1.6) and (1.7) are reciprocal to each other. This fact combined with Lemma 1 enables us to strengthen the condition (1.4) for the $D$-space as follows.

**Lemma 2.** *Let $(X, m, \mathcal{F}, \mathcal{E})$ be a D-space. If $u_1, u_2, \cdots, u_n\in\mathcal{F}$ and if a measurable function $w$ on $X$ satisfies $|w(x)|\leqq\sum_{i=1}^n|u_i(x)|$, $|w(x)-w(y)|\leqq\sum_{i=1}^n|u_i(x)-u_i(y)|$ $m$-a.e, then $w\in\mathcal{F}$ and $\sqrt{\mathcal{E}^\alpha(w,w)}\leqq\sum_{i=1}^n\sqrt{\mathcal{E}^\alpha(u_i,u_i)}$, $\alpha\geqq 0$, $\mathcal{E}^0$ standing for $\mathcal{E}$.*

## §2. Dirichlet rings and equivalence of Dirichlet spaces.

Consider a $D$-space $(X, m, \mathcal{F}, \mathcal{E})$. We set, for $u\in L^\infty(X)$ $(=L^\infty(X; m))$, $\|u\|_\infty=\text{ess-sup}_{x\in X}|u(x)|$. Let us put

$$\mathcal{F}^{(b)}=\mathcal{F}\cap L^\infty(X), \tag{2.1}$$

$$|||u|||_\alpha=\sqrt{\mathcal{E}^\alpha(u,u)}+\|u\|_\infty,\qquad u\in\mathcal{F}^{(b)},\qquad \alpha>0. \tag{2.2}$$

**Theorem 1.** (i) *For each $\alpha>0$, $(\mathcal{F}^{(b)}, |||\ \ |||_\alpha)$ is a normed ring, two functions of $\mathcal{F}^{(b)}$ being identified if they coincide $m$-a.e. Exactly speaking, it is a complex Banach space and, for any $u, v\in\mathcal{F}^{(b)}$, $u\cdot v\in\mathcal{F}^{(b)}$ and $|||u\cdot v|||_\alpha\leqq|||u|||_\alpha\cdot|||v|||_\alpha$.*

(ii) *For any $u\in\mathcal{F}^{(b)}$ and $\alpha>0$,*

$$\lim_{n\to+\infty}\sqrt[n]{|||u^n|||_\alpha}=\|u\|_\infty. \tag{2.3}$$

**Proof.** Applying Lemma 2 to functions $w=u\cdot v$, $u_1=B\cdot u$ and $u_2=A\cdot v$ with $A=\|u\|_\infty$, $B=\|v\|_\infty$, we see that $w\in\mathcal{F}$ and $\sqrt{\mathcal{E}^\alpha(w,w)}\leqq B\sqrt{\mathcal{E}^\alpha(u,u)}+A\sqrt{\mathcal{E}^\alpha(v,v)}$. This implies the latter assertion of (i). The left hand side of (2.3) is not greater than $A=\|u\|_\infty$, since $u^n$ is a normal contraction of $nA^{n-1}u$. The converse inequality is trivial.

We call $(\mathcal{F}^{(b)}, ||| \ |||_\alpha, \alpha>0)$ the *Dirichlet ring* (in short, *D-ring*) induced by $(X, m, \mathcal{F}, \mathcal{E})$. This ring has not necessarily a unit element for multiplication.

**Lemma 3.** *Let $\mathcal{F}_1$ be a Dirichlet subspace of $\mathcal{F}$ and $L$ be a closed subring of $(L^\infty(X), \| \ \|_\infty)$. We assume that $u\in L$ implies $\bar{u}\in L$. Then, the intersection $\mathcal{R}$ of $\mathcal{F}_1$ and $L$ is a closed subring of $(\mathcal{F}^{(b)}, ||| \ |||_\alpha)$. $\mathcal{R}$ is semi-simple and symmetric with respect to the operation of taking complex conjugate function. $\mathcal{R}$ is a function lattice; for any real $u, v\in\mathcal{R}$, $u\vee v$ and $u\wedge v$ are also in $\mathcal{R}$. Further, for any real $u\in\mathcal{R}$, $u\wedge 1\in\mathcal{R}$.*

**Proof.** In view of the equality (2.3), $\mathcal{R}$ is semi-simple. $\mathcal{R}$ is symmetric, or equivalently ([4]), $u\in\mathcal{R}$ implies $v=|u|^2/1+|u|^2\in\mathcal{R}$, because $v$ is a normal contraction of $|u|^2\in\mathcal{R}$. For real $u\in\mathcal{R}$, $|u|$ and $u\wedge 1$ are normal contractions of $u$, yielding the final statement.

We will call two $D$-spaces $(X, m, \mathcal{F}, \mathcal{E})$ and $(\tilde{X}, \tilde{m}, \tilde{\mathcal{F}}, \tilde{\mathcal{E}})$ *equivalent* iff their associated $D$-rings $(\mathcal{F}^{(b)}, ||| \ |||_\alpha, \alpha>0)$ and $(\tilde{\mathcal{F}}^{(b)}, |||\tilde{\ }|||_\alpha, \alpha>0)$ are isomorphic and isometric, exactly speaking, iff there is a ring isomorph $\Phi$ from $\mathcal{F}^{(b)}$ onto $\tilde{\mathcal{F}}^{(b)}$ and $|||\widetilde{\Phi u}|||_\alpha=|||u|||_\alpha$ for any $u\in\mathcal{F}^{(b)}$ and $\alpha>0$.

**Theorem 2.** *Suppose that D-spaces $(X, m, \mathcal{F}, \mathcal{E})$ and $(\tilde{X}, \tilde{m}, \tilde{\mathcal{F}}, \tilde{\mathcal{E}})$ are equivalent under a mapping $\Phi$ from $\mathcal{F}^{(b)}$ onto $\tilde{\mathcal{F}}^{(b)}$. Then, $\Phi$ turns out to be a lattice isomorph and $\Phi$ can be extended uniquely to the next kinds of transformations.*

(a) *A unitary mapping $\Phi_1$ from $(\mathcal{F}, \mathcal{E})$ onto $(\tilde{\mathcal{F}}, \tilde{\mathcal{E}})$,*

(b) *A unitary mapping $\Phi_2$ from $L_0^2(X)$ onto $L_0^2(\tilde{X})$,*

(c) *A ring isomorphic and isometric mapping $\Phi_3$ from $L_0^\infty(X)$ onto $L_0^\infty(\tilde{X})$. Here, $L_0^2(X)(L_0^\infty(X))$ is the closure of $\mathcal{F}(\mathcal{F}^{(b)})$ in the metric space $L^2(X)(L^\infty(X))$. $L_0^2(\tilde{X})$ and $L_0^\infty(\tilde{X})$ are defined in the same way. Further, the associated $L^2$-resolvents $\{G_\alpha, \alpha>0\}$ and $\{\tilde{G}_\alpha, \alpha>0\}$ are related by*

$$(2.4)\qquad \tilde{G}_\alpha\tilde{u}=\Phi_2 G_\alpha \Phi_2^{-1}\tilde{u},\qquad \tilde{u}\in L_0^2(\tilde{X}),\qquad \alpha>0.$$

**Proof.** Owing to the equality (2.3), $\Phi$ preserves the uniform norm. On the other hand, $\mathcal{F}^{(b)}$ is dense in $\mathcal{F}$ with metric $\mathcal{E}^\alpha$ ([2]; Lemma 2.1 and Theorem 2.1 (iii)). All the assertions but (2.4) follow from these facts and the definition of equivalence. Take $\tilde{u}\in L_0^2(\tilde{X})$. By (1.5), we have for any $\tilde{v}\in\tilde{\mathcal{F}}$,

$$\tilde{\mathcal{E}}^\alpha(\tilde{G}_\alpha\tilde{u}, \tilde{v})=(\tilde{u}, \tilde{v})_{\tilde{X}}=(\Phi_2^{-1}\tilde{u}, \Phi_2^{-1}\tilde{v})_X=\mathcal{E}^\alpha(G_\alpha\Phi_2^{-1}\tilde{u}, \Phi_2^{-1}\tilde{v})$$
$$=\tilde{\mathcal{E}}^\alpha(\Phi_2 G_\alpha\Phi_2^{-1}\tilde{u}, \tilde{v}),$$

which implies (2.4).

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