# 135. *The Subordination of Lévy System for Markov Processes*

By Akihiko MIYAKE

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**§1. Preliminary notions and the result.** For each process $x(t)$ belonging to a certain class of Markov processes, the Lévy measure $n(x, dy)$ is defined as follows [1]:

$$(1)\qquad \lim_{t\to+0} T_t f(x)/t = \lim_{t\to+0}\int_S f(y)P(t, x, dy)/t = \int_S f(y)n(x, dy) \qquad \text{for every } x \in D$$

where $S$ and $\hat{S}$, are respectively, a locally compact Hausdorff space satisfying the 2nd axiom of countability and its one-point compactification, $D$ is a bounded open set in $S$, and $f$ is a function in $C(\hat{S})$ whose support does not intersect $D$. $\{T_t\}$ and $\{P(t, x, dy)\}$ respectively, are the semigroup and the transition functions of the process $x(t)$, and the convergence in (1) is a bounded convergence in $D$.

We know that, when the time of such a Markov process is changed by a temporally homogeneous non-decreasing Lévy process $h(t)$ which is independent of $x(t)$ and has the Lévy measure $\dot{n}(t)$:

$$(2)\qquad \dot{E}e^{rh(t)} = \exp\left[-t\left\{cr + \int_0^\infty (1-e^{-ru})\dot{n}(du)\right\}\right]$$
$$c \geq 0, \quad \int_0^\infty \frac{u}{1+u}\dot{n}(du) < \infty,$$

then the Lévy measure $\tilde{n}(x, dy)$ of the new Markov process is as follows [1]:

$$(3)\qquad \tilde{n}(x, dy) = cn(x, dy) + \int_0^\infty P(t, x, dy)\dot{n}(dt).$$

Furthermore, for each process $x(t)$ belonging to a wider class of Markov processes, that is, the class of Hunt processes with reference measures on $S$, the Lévy system $(n(x, dy), A)$, the pair of a kernel $n(x, dy)$ and an additive functional $A(t)$ of $x(t)$, is defined as a generalization of the Lévy measure defined above as follows [2]:

$$(4)\qquad E_x \sum_{s\leqq t} f(x(s-), x(s)) = E_x\left[\int_0^t \left\{\int_{\hat{S}} f(x(s), y)\, n(x(s), dy)\right\} dA(s)\right]$$

where $f$ is an $F(S\times\hat{S})$-measurable non-negative function such that $f(x, x)=0$ for any $x \in S$, and $F(S\times\hat{S})$ is the completion of the topological Borel field on $S\times\hat{S}$ with respect to the family of all bounded measures. If $A(t)$ is the minimum of $t$ and the life time of $x(t)$, then the kernel $n(x, dy)$ of the Lévy system coincides with the Lévy measure in (1).

The purpose of the present paper is to prove the following

**Theorem.** *Let $y(t)$ be the new process $x(h(t))$ obtained by subordinating $x(t)$ by $h(t)$, $\tilde{B}(t)$ the minimum of $t$ and the life time of $y(t)$, and let $\tilde{A}^c(t)$ be the continuous part of $A(h(t))$.*

*Then the Lévy system $(n(x, dy), A)$ is changed as follows:*

$$(5)\quad \tilde{E}_x \sum_{s\le t} f(y(s-), y(s))$$
$$= \tilde{E}_x\left[\int_0^t \int_{\hat{S}} f(y(s), y)\left\{n(y(s), dy)\, d\tilde{A}^c(s) + \int_0^\infty P(q, y(s), dy)\dot{n}(dq) d\tilde{B}(s)\right\}\right]$$

*and $\tilde{B}(t)$ and $\tilde{A}^c(t)$ are addivive functionals of $y(t)$.*

**§2. Proof of Theorem.** Let the event spaces of $x(t)$, $h(t)$ and $y(t)$ be $W$, $\dot{W}$ and $\tilde{W} = W \times \dot{W}$ respectively.

In the case $c=0$ and $n((0, \infty)) < \infty$ in (2), paths of $h(t)$ being step functions, we have

$$E = \tilde{E}_x \sum_{s\le t} f(y(s-), y(s))$$
$$= \dot{E} E_x \sum_{s\in I(t,\dot{w})} f(x(h(s-, \dot{w}), w), x(h(s, \dot{w}), w))$$

where $I(t, \dot{w}) = \{s;\ h(s, \dot{w}) - h(s-, \dot{w}) > 0,\ s \in [0, t]\}$. Since $I(t, \dot{w})$ is countable,

$$E = \dot{E} \sum_{s\le t} E_x f(x(h(s-, \dot{w}), w), x(h(s, \dot{w}), w)).$$

As the function $g(s, t) = E_x f(x(s, w), x(t, w))$ satisfys $g(s, s) = 0$ for any $s$, we can apply the property (4) of the Lévy measure $\dot{n}(s, dt) = \dot{n}(d(t-s))$ of $h(t)$, and obtain

$$(6)\qquad E = \dot{E}\left[\int_0^t \left\{\int_0^\infty [E_x f(x(h(s, \dot{w}), w)\, x(r, w))]\dot{n}(h(s, \dot{w}), dr)\right\} ds\right].$$

From the Markov property of $x(t)$,

$$E = \dot{E}\left[\int_0^t \left\{\int_0^\infty E_x[\ddot{E}_{x(h(s,\dot{w}),w)} f(x(0, \ddot{w}), x(r - h(s, \dot{w}), \ddot{w}))]\dot{n}(h(s, \dot{w}), dr)\right\} ds\right]$$
$$= \dot{E}\left[\int_0^t \left\{\int_0^\infty E_x[\ddot{E}_{x(h(s,\dot{w})w)} f(x(0, \ddot{w}), x(q, \ddot{w}))]\, \dot{n}(dq)\right\} ds\right]$$
$$= \dot{E}\left[\int_0^t \left\{\int_0^\infty E_x\left[\int_{\hat{S}} f(x(h(s, \dot{w}), w), y)\, P(q, x(h(s, \dot{w}), w), dy)\right] \dot{n}(dq)\right\} ds\right]$$
$$= \dot{E}\left[\int_0^t E_x\left[\int_{\hat{S}} f(x(h(s, \dot{w}), w), y)\left\{\int_0^\infty P(q, x(h(s, \dot{w}), w), dy)\dot{n}(dq)\right\}\right] ds\right]$$
$$= \dot{E}\left[E_x\left\{\int_0^t \left[\int_{\hat{S}} f(x(h(s, \dot{w}), w), y)\left\{\int_0^\infty P(q, x(h(s, \dot{w}), w), dy)\dot{n}(dq)\right\}\right]\right.\right.$$
$$\left.\left. d\tilde{B}(s, (w, \dot{w}))\right\}\right]$$
$$= \tilde{E}_x\left[\int_0^t \left\{\int_{\hat{S}} f(y(s, \tilde{w}), y)\left\{\int_0^\infty P(q, y(s, \tilde{w}), dy)\dot{n}(dq)\right\}\right\} d\tilde{B}(s, \tilde{w})\right]$$

where $\ddot{E}_x$ is the integral with respect to the measure $P_x(d\ddot{w})$, $\tilde{B}(t, \tilde{w}) = \tilde{B}(t, (w, \dot{w})) = \min(t, \tilde{\xi}(\tilde{w}))$ and $\tilde{\xi}(\tilde{w}) = \inf\{t;\ \xi(w) \le h(t, \dot{w})\}$; here $\xi(w)$

denotes the life time of $x(t)$, accordingly $\tilde{\xi}(\tilde{w})$ is the life time of $y(t) = x(h(t))$.

In the case $c \neq 0$ or $n((0, \infty)) = \infty$ in (2), the discontinuous points of $y(t, \tilde{w}) = x(h(t, \dot{w}), w)$ are determined by the discontinuous points of $h(t, \dot{w})$ and those of $x(t, w)$. Hence

$$\tilde{E}_x \sum_{s \leq t} f(y(s-), y(s)) = E_1 + E_2$$

$$E_1 = \dot{E} E_x \sum_{s \in I(t, \dot{w})} f(x(h(s-, \dot{w})-, w), x(h(s, \dot{w}), w))$$

$$E_2 = \dot{E} E_x \sum_{s \in [0, t] - I(t, \dot{w})} f(x(h(s, \dot{w})-, w), x(h(s, \dot{w}), w)).$$

Since $I(t, \dot{w})$ is countable and $x(t)$ has no fixed discontinuous points,

$$E_1 = \dot{E} \sum_{s \leq t} E_x f(x(h(s-, \dot{w}), w), x(h(s, \dot{w}), w)).$$

In the same way as in the first case, we obtain

$$E_1 = \tilde{E}_x \left[ \int_0^t \left\{ \int_{\hat{S}} f(y(s, \tilde{w}), y) \left\{ \int_0^\infty P(q, y(s, \tilde{w}), dy)\, \dot{n}(dq) \right\} \right\} d\tilde{B}(s, \tilde{w}) \right].$$

Next, we notice that $E_2$ can be written as

$$\dot{E} E_x \sum_{r \in \Gamma(t, \dot{w})} f(x(r-, w), x(r, w))$$

where $\Gamma(t, \dot{w}) = \{h(s, \dot{w}); s \in [0, t] - I(t, \dot{w})\}$. Then from the property (4) of the Lévy system of $x(t)$, we have

$$E_2 = \dot{E}\left[ E_x \left[ \int_{\Gamma(t, \dot{w})} \left\{ \int_{\hat{S}} f(x(r, w), y)\, n(x(r, w), dy) \right\} dA(r, w) \right] \right]$$

$$= \dot{E}\left[ E_x \left[ \int_{[0, t] - I(t, \dot{w})} \left\{ \int_{\hat{S}} f(x(h(s, \dot{w}), w), y) n(x(h(s, \dot{w}), w), dy) \right\} dA(h(s, \dot{w}), w) \right] \right]$$

Put $\tilde{A}(t, \tilde{w}) = A(h(t, \dot{w}), w)$. Then it is an additive functional of $y(t, w)$. In fact, since the shift in $\tilde{W}$ is defined by

$$\tilde{w}_t^+ = (w, \dot{w})_t^+ = (w_{h(t, \dot{w})}^+, \ddot{w}) \text{ and } h(s, \ddot{w}) = h(t+s, \dot{w}) - h(t, \dot{w}),$$

the additivity of $\tilde{A}(t, \tilde{w})$ is derived as follows;

$$\begin{aligned} \tilde{A}(t+s) &= A(h(t+s, \dot{w}), w) \\ &= A(h(t, \dot{w}), w) + A(h(t+s, \dot{w}) - h(t, \dot{w}), w_{h(t, \dot{w})}^+) \\ &= \tilde{A}(t, \tilde{w}) + \tilde{A}(s, \tilde{w}_t^+). \end{aligned}$$

Furthermore the continuous part $\tilde{A}^c(t, \tilde{w})$ of $\tilde{A}(t, \tilde{w})$ defined by

$$\tilde{A}^c(t, \tilde{w}) = \tilde{A}(t, \tilde{w}) - \sum_{s \leq t} \{A(s, \tilde{w}) - A(s-, \tilde{w})\}$$

is also an additive functional of $y(t, \tilde{w})$; it vanishes when $c = 0$ in (2). Therefore

$$E_2 = \dot{E}\left[ E_x \left[ \int_0^t \left\{ \int_{\hat{S}} f(x(h(s, \dot{w}), w), y) n(x(h(s, \dot{w}), w), dy) \right\} d\tilde{A}^c(s, (w, \dot{w})) \right] \right]$$

$$= \tilde{E}_x \left[ \int_0^t \left\{ \int_{\hat{S}} f(y(s, \tilde{w}), y) n(y(s, \tilde{w}), dy) \right\} d\tilde{A}^c(s, \tilde{w}) \right].$$

Taking account of the fact that $\tilde{A}^c$ is identically zero in the case $c = 0$, we may conclude that $E_1 + E_2$ equals the right side of (5). Thus the proof is completed.

In particular, if $A(t, w)=\min(t, \xi(w))$, then we have

$$\tilde{A}^c(t, \tilde{w})=c\tilde{B}(t, \tilde{w})$$

and

$$\tilde{E}_x \sum_{s\leq t} f(y(s-), y(s))$$
$$=\tilde{E}_x\left[\int_0^t\left[\int_{\hat{S}} f(y(s), y)\left\{cn(y(s), dy)+\int_0^\infty P(q, y(s), dy)\dot{n}(dq)\right\}\right] d\tilde{B}(s)\right].$$

The last equality implies (3).

## References


[1] N. Ikeda and S. Watanabe: On some ralations between the harmonic measure and the Lévy measure for a certain class of Markov processes. J. Math. Kyoto Univ., **2**, 79–95 (1962).

[2] S. Watanabe: On discontinuous additive functionals and Lévy measure of a Markov process. J. Math. Soc. Japan, **14**, 53–70 (1964).