130. *5-dimensional Orientable Submanifolds of R^7. I*

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Introduction. It is well known that the odd dimensional number space R^{2n+1}, the odd dimensional sphere S^{2n+1} and orientable hypersurfaces of an almost complex manifold etc. admit an almost contact structure.

The main purpose of this paper is to show that, using the vector cross product induced by Cayley numbers, any 5-dimensional orientable submanifold of any 7-dimensional parallelizable manifold admits an almost contact structure.

2. Basic informations.

(a) Almost contact manifolds.

An almost contact structure (ϕ, ξ, η) on a $(2n+1)$-dimensional C^∞ manifold M is given by a tensor field ϕ of type (1, 1), a vector field ξ and a 1-form η called the *contact form* such that

$$\eta(\xi)=1, \tag{1}$$

$$\phi(\xi)=0, \quad \eta\circ\phi=0, \tag{2}$$

$$\phi^2=-I+\eta(\cdot)\xi, \tag{3}$$

where I is the identity transformation field.

If M has a (ϕ, ξ, η)-structure then we can find a Riemannian metric $\langle\ ,\ \rangle$ such that

$$\eta=\langle\xi, \cdot\rangle, \tag{4}$$

$$\langle\phi X, \phi Y\rangle=\langle X, Y\rangle-\eta(X)\eta(Y), \tag{5}$$

for any vector fields X, Y on M, so that ϕ is skew symmetric with respect to $\langle\ ,\ \rangle$. M is then said to have a $(\phi, \xi, \eta, \langle\ ,\ \rangle)$-structure and $\langle\ ,\ \rangle$ is called the *associated Riemannian metric* of (ϕ, ξ, η).

(b) Vector cross products on certain 7-dimensional Riemannian manifolds.

The *vector cross product* on a certain 7-dimensional Riemannian manifold $\bar{M}$ is a linear map $P: V(\bar{M})\times V(\bar{M})\to V(\bar{M})$ (writing here $P(\bar{X}, \bar{Y})=\bar{X}\otimes\bar{Y}$) satisfying the following conditions:

$$\bar{X}\otimes\bar{Y}=-\bar{Y}\otimes\bar{X}, \tag{6}$$

$$\langle\bar{X}\otimes\bar{Y}, \bar{Z}\rangle=\langle\bar{X}, \bar{Y}\otimes\bar{Z}\rangle, \tag{7}$$

$$(\bar{X}\otimes\bar{Y})\otimes\bar{Z}+\bar{X}\otimes(\bar{Y}\otimes\bar{Z})=2\langle\bar{X}, \bar{Z}\rangle\bar{Y}-\langle\bar{Y}, \bar{Z}\rangle\bar{X}-\langle\bar{X}, \bar{Y}\rangle\bar{Z}, \tag{8}$$

where $V(\bar{M})$ denotes the ring of differentiable vector fields on $\bar{M}$ and $\bar{X}, \bar{Y}, \bar{Z}\in V(\bar{M})$. Any parallelizable 7-dimensional Riemannian manifold, for example, R^7, 7-dimensional hyperbolic space, 7-dimensional sphere S^7 and $U(4)/U(3)$ admit such vector cross product. Moreover, for R^7, the tensor P is parallel so that

$$\bar{\nabla}_{\bar{X}}(\bar{Y}\otimes\bar{Z})=\bar{\nabla}_{\bar{X}}\bar{Y}\otimes\bar{Z}+\bar{Y}\otimes\bar{\nabla}_{\bar{X}}\bar{Z} \tag{9}$$

holds good, where $\bar{\nabla}$ denotes the covariant differentiation of R^7.

The dimensions of such $\bar{M}$ admitting above vector cross product are essentially only three and seven ([1]).

3. 5-dimensional orientable submanifolds of R^7.

Let M be a 5-dimensional orientable submanifold of R^7. Then there exist locally defined mutually orthogonal differentiable unit normal vector fields C_1, C_2 to M.

Proposition 1. *$C_1\otimes C_2$ is independent of the choice of mutually orthogonal differentiable unit normal vector fields C_1, C_2 to M.*

Proof. Let $\bar{C}_1, \bar{C}_2$ be another mutually orthogonal unit normal vector fields to M. Then we may write $\bar{C}_1, \bar{C}_2$ as a linear combinations of C_1, C_2:

$$\begin{cases}\bar{C}_1=aC_1+bC_2\\ \bar{C}_2=cC_1+dC_2, \quad \text{with } ad-bc=1,\end{cases}$$

where a, b, c and d are differentiable functions.

Then we have

$$\begin{aligned}\bar{C}_1\otimes\bar{C}_2&=(aC_1+bC_2)\otimes(cC_1+dC_2)\\ &=adC_1\otimes C_2+bcC_2\otimes C_1\\ &=(ad-bc)C_1\otimes C_2\\ &=C_1\otimes C_2.\end{aligned}$$

Q.E.D.

Proposition 2. *For mutually orthogonal unit normal vector fields C_1, C_2 to M, we have*

$$C_1\otimes C_2 \text{ is tangent to } M, \tag{10}$$

$$\|C_1\otimes C_2\|=1. \tag{11}$$

And, for $X\in V(M)$, we have

$$X\otimes(C_1\otimes C_2) \text{ is tangent to } M. \tag{12}$$

Proof. For (10), we have

$$\begin{aligned}\langle C_1\otimes C_2, C_1\rangle&=-\langle C_2\otimes C_1, C_1\rangle \quad \text{(by (6))}\\ &=-\langle C_2, C_1\otimes C_1\rangle \quad \text{(by (7))}\\ &=0.\end{aligned}$$

Similarly we have $\langle C_1\otimes C_2, C_2\rangle=0$. Consequently, $C_1\otimes C_2$ is tangent to M. For (11), we have

$$\begin{aligned}\langle C_1\otimes C_2, C_1\otimes C_2\rangle&=\langle C_1, C_2\otimes(C_1\otimes C_2)\rangle\\ &=-\langle C_1, C_2\otimes(C_2\otimes C_1)\rangle\\ &=-\langle C_1, (C_2\otimes C_2)\otimes C_1+C_2\otimes(C_2\otimes C_1)\rangle\\ &=-\langle C_1, 2\langle C_2, C_1\rangle C_2-\langle C_2, C_1\rangle C_2-\langle C_2, C_2\rangle C_1\rangle\\ &=-\langle C_1, -\langle C_2, C_2\rangle C_1\rangle\\ &=1.\end{aligned}$$

Finally for (12), we have

$$\begin{aligned}\langle X\otimes(C_1\otimes C_2),\, C_1\rangle &=\langle X,\, (C_1\otimes C_2)\otimes C_1\rangle\\ &=-\langle X,\, (C_2\otimes C_1)\otimes C_1\rangle\\ &=-\langle X,\, (C_2\otimes C_1)\otimes C_1+C_2\otimes(C_1\otimes C_1)\rangle\\ &=-\langle X,\, 2\langle C_2, C_1\rangle C_1-\langle C_1, C_1\rangle C_2-\langle C_2, C_1\rangle C_1\rangle\\ &=0. \qquad\qquad \text{Q.E.D.}\end{aligned}$$

Theorem 1. *A 5-dimensional orientable submanifold M of R^7 admits an almost contact structure so that M is an almost contact manifold.*

Proof. By above Propositions 1 and 2, we can define ϕ, ξ and η as follows:

(13) $$\xi=C_1\otimes C_2,$$

(14) $$\eta(X)=\langle C_1\otimes C_2,\, X\rangle,$$

(15) $$\phi(X)=X\otimes(C_1\otimes C_2).$$

Then (ϕ, ξ, η) gives an almost contact structure on M.

In fact, we have

$$\begin{aligned}\eta(\xi)&=\langle C_1\otimes C_2,\, C_1\otimes C_2\rangle=1,\\ \phi(\xi)&=(C_1\otimes C_2)\otimes(C_1\otimes C_2)=0,\\ (\eta\circ\phi)(X)&=\eta(X\otimes(C_1\otimes C_2))\\ &=\langle C_1\otimes C_2,\, X\otimes(C_1\otimes C_2\rangle\\ &=\langle X,\, (C_1\otimes C_2)\otimes(C_1\otimes C_2)\rangle\\ &=0.\end{aligned}$$

Finally, we have

$$\begin{aligned}\phi^2(X)&=\phi(\phi X)\\ &=\phi(X\otimes(C_1\otimes C_2))\\ &=(X\otimes(C_1\otimes C_2))\otimes(C_1\otimes C_2)+X\otimes((C_1\otimes C_2)\otimes(C_1\otimes C_2))\\ &=2\langle X,\, C_1\otimes C_2\rangle C_1\otimes C_2-\langle C_1\otimes C_2,\, C_1\otimes C_2\rangle X-\langle X,\, C_1\otimes C_2\rangle C_1\otimes C_2\\ &=-X+\langle X,\, C_1\otimes C_2\rangle C_1\otimes C_2\\ &=-X+\eta(X)\xi. \qquad\qquad \text{Q.E.D.}\end{aligned}$$

Remark 1. M is an almost contact metric manifold $(\phi, \xi, \eta, \langle\ ,\ \rangle)$ with the induced Riemannian metric on M as an associated Riemannian metric of (ϕ, ξ, η). Indeed, we have

$$\begin{aligned}\langle \phi X,\, \phi Y\rangle&=\langle X\otimes(C_1\otimes C_2),\, Y\otimes(C_1\otimes C_2)\rangle\\ &=\langle X,\, (C_1\otimes C_2)\otimes(Y\otimes(C_1\otimes C_2))\rangle\\ &=-\langle X,\, (C_1\otimes C_2)\otimes((C_1\otimes C_2)\otimes Y)\rangle\\ &=-\langle X,\, (C_1\otimes C_2)\otimes((C_1\otimes C_2)\otimes Y)+((C_1\otimes C_2)\otimes(C_1\otimes C_2))\otimes Y\rangle\\ &=-\langle X,\, 2\langle C_1\otimes C_2,\, Y\rangle C_1\otimes C_2-\langle C_1\otimes C_2,\, Y\rangle C_1\otimes C_2\\ &\quad-\langle C_1\otimes C_2,\, C_1\otimes C_2\rangle Y\rangle\\ &=-\langle X,\, \langle C_1\otimes C_2,\, Y\rangle C_1\otimes C_2-Y\rangle\\ &=\langle X,\, Y\rangle-\eta(X)\eta(Y).\end{aligned}$$

Remark 2. If we put $\bar{\xi}=a\xi$, $\bar{\eta}=1/a\eta$, where a is a non-zero constant, then $(\phi, \bar{\xi}, \bar{\eta})$ also gives an almost contact structure on M.

Remark 3. For an orientable hypersurface L of R^7, we set $JX = X \otimes C$, where X is a tangent and C is a normal vector field to L. Then J gives an almost complex structure on L. A. Gray ([1]) treated this case in detail.

Through the proof of Theorem 1, we did not use the property of (9). Thus considering M as an orientable submanifold of any parallelizable 7-dimensional manifold, we have

Corollary 1. *Any 5-dimensional orientable submanifold of any 7-dimensional parallelizable Riemannian manifold admits an almost contact structure.*

Using the result of [3], we have

Corollary 2. *The direct product of any two 5-dimensional orientable submanifolds of any parallelizable 7-dimensional Riemannian manifold admits an almost complex structure.*

References

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