# 128. On Some Properties of $A^p(G)$-algebras*)

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**1. Introduction.** Let $G$ be a locally compact abelian group with dual group $\hat{G}$. We denote $dx$ and $d\hat{x}$ the Haar measures of $G$ and $\hat{G}$ respectively. Recently, Larsen, Liu, and Wang [4] have investigated a space $A^p(G)$ consisting of all complex-valued functions $f \in L^1(G)$ whose Fourier transforms $\hat{f}$ belong to $L^p(\hat{G})$ $(p \geqslant 1)$. In this paper, we shall show further investigations of the algebra $A^p(G)$ proving the existence of the approximate identities of $A^p(G)$ and using the approximate identity to give a reproof of Theorem 5 in [4]. We show also that the closed primary ideal of $A^p(G)$ is maximal.

**2. The approximate identities of $A^p(G)$-algebras.** It is clear that $A^p(G)$ is an ideal dense in $L^1(G)$ under convolution. Indeed, for any $f \in A^p(G)$ and $g \in L^1(G)$,

$$\| \widehat{f*g} \|_p \leqslant \| \hat{g} \|_\infty \| \hat{f} \|_p$$

proving $f*g \in A^p(G)$ and the density of $A^p(G)$ in $L^1(G)$ follows from the fact that if $\{e_\alpha\}$ is an approximate identity in $L^1(G)$ whose Fourier transforms have compact supports then $e_\alpha \in A^p(G)$ and for an arbitrary function $f \in L^1(G)$ we have

$$f*e_\alpha \in A^p(G) \text{ and } \| f*e_\alpha - f \|_1 \to 0.$$

Define the norm of $f \in A^p(G)$ $(1 \leqslant p < \infty)$ by

$$\| f \|^p = \| f \|_1 + \| \hat{f} \|_p$$

where $\| f \|_1 = \int_G | f(x) | \, dx$ and $\| \hat{f} \|_p = \left( \int_{\hat{G}} | \hat{f}(\hat{x}) |^p \, d\hat{x} \right)^{1/p}$. Then $A^p(G)$ is a commutative Banach algebra under convolution as its product and with the norm $\| \cdot \|^p$ (see [4; Theorem 3]).

We say here an approximate identity for $A^p(G)$ a family $\{e_\alpha\}$ of functions in $A^p(G)$ such that for any $f \in A^p(G)$ and $\varepsilon > 0$, there exists $e_\alpha \in \{e_\alpha\}$ such that $\| e_\alpha * f - f \|^p < \varepsilon$.

**Theorem 1.** *The Banach algebra $A^p(G)$ has an approximate identity with the properties that it is also the bounded approximate identity for $L^1(G)$ and whose Fourier transform has compact support in $\hat{G}$.*

**Proof.** By Rudin [7] Theorem 2.6.6, we see that there is a

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bounded approximate identity $\{e_\alpha\}$ in $L^1(G)$ such that each $\hat{e}_\alpha$ has compact support in $\hat{G}$ where $\{\alpha\}$ is a directed set. Let $K$ be an arbitrary compact set in $\hat{G}$. Then there exists a function $k \in L^1(G)$ such that $\hat{k}=1$ on $K$. Thus

$$\widehat{e_\alpha * k}(\hat{x}) = \hat{e}_\alpha(\hat{x})\hat{k}(\hat{x}) = \hat{e}_\alpha(\hat{x})$$

for any $\hat{x} \in K$. Now for a given $\varepsilon > 0$ there exists an index $\alpha_0$ such that $\| e_\alpha * k - k \|_1 < \varepsilon$ whenever $\alpha > \alpha_0$. Then for any $\hat{x} \in K$,

$$\begin{aligned} | \hat{e}_\alpha(\hat{x}) - 1 | &= | \hat{e}_\alpha(\hat{x})\hat{k}(\hat{x}) - \hat{k}(\hat{x}) | \leqslant \| \hat{e}_\alpha \hat{k} - \hat{k} \|_\infty \\ &\leqslant \| e_\alpha * k - k \|_1 < \varepsilon \end{aligned}$$

for $\alpha > \alpha_0$. Hence $\hat{e}_\alpha$ converges to 1 uniformly on any compact set in $\hat{G}$. We assert that $\{e_\alpha\}$ becomes an approximate identity for $A^p(G)$ as follows.

Since $f \in A^p(G)$, $\hat{f} \in L^p(\hat{G})$. Therefore for a given $\varepsilon > 0$, we may choose a compact set $K = K_\varepsilon$ in $\hat{G}$ so that

$$\int_{\sim K} | \hat{f}(\hat{x}) |^p d\hat{x} < \varepsilon^p / 2^{2p+1} M^p$$

where $\sim K$ is the complement of the set $K$ and $M$ is a constant such that $\| e_\alpha \|_1 \leqslant M$. As $\hat{e}_\alpha \to 1$ uniformly on $K$,

$$\int_K | \hat{f}(\hat{x}) \hat{e}_\alpha(\hat{x}) - \hat{f}(\hat{x}) |^p d\hat{x} \to 0.$$

Thus there exists $\alpha_0$ such that

$$\int_K | \hat{f}(\hat{x}) \hat{e}_\alpha(\hat{x}) - \hat{f}(\hat{x}) |^p \, d\hat{x} < \frac{\varepsilon^p}{2^{p+1}}$$

for $\alpha > \alpha_0$ and so

$$\begin{aligned} &\int_{\hat{G}} | \hat{f}(\hat{x}) \hat{e}_\alpha(\hat{x}) - \hat{f}(\hat{x}) |^p \, d\hat{x} \\ &= \int_K | \hat{f}(\hat{x}) \hat{e}_\alpha(\hat{x}) - \hat{f}(\hat{x}) |^p \, d\hat{x} + \int_{\sim K} | \hat{f}(\hat{x}) \hat{e}_\alpha(\hat{x}) - \hat{f}(\hat{x}) |^p \, d\hat{x} \\ &\leqslant \int_K \hat{f}(\hat{x}) \hat{e}_\alpha(\hat{x}) - \hat{f}(\hat{x}) |^p \, d\hat{x} + 2^p M^p \int_{\sim K} | \hat{f}(\hat{x}) |^p \, d\hat{x} \\ &< \varepsilon^p / 2^{p+1} + \varepsilon^p / 2^{p+1} = \varepsilon^p / 2^p \end{aligned}$$

whenever $\alpha > \alpha_0$. Therefore

$$\| \widehat{f * e_\alpha} - \hat{f} \|_p < \varepsilon / 2 \qquad \text{for } \alpha > \alpha_0.$$

On the other hand, since $\{e_\alpha\}$ is an approximate identity for $L^1(G)$, there is an index $\alpha_1$ such that

$$\| f * e_\alpha - f \|_1 < \varepsilon / 2 \qquad \text{for } \alpha > \alpha_1.$$

Letting $\alpha_2 = \sup(\alpha_0, \alpha_1)$, we obtain

$$\begin{aligned} \| f * e_\alpha - f \|^p &= \| f * e_\alpha - f \|_1 + \| \widehat{f * e_\alpha} - \hat{f} \|_p \\ &< \varepsilon/2 + \varepsilon/2 = \varepsilon \end{aligned}$$

whenever $\alpha > \alpha_2$. This completes the proof. Q.E.D.

We notice that contrary to the usual case the approximate identity in $A^p(G)$ can not be chosen to be uniformly bounded in general. Indeed, let $\{e_\alpha\}$ be an arbitrary approximate identity for $A^p(G)$, then $\{e_\alpha\}$ is also an approximate identity for $L^1(G)$. Consider a compact set $K$ with positive measure. As we have shown before, $\hat{e}_\alpha \to 1$ uniformly on $K$. Hence for any $\varepsilon > 0$ there exists $\alpha$ such that

$$\int_K (|\hat{e}_\alpha(\hat{x})|^p - 1) d\hat{x} > -\varepsilon$$

or

$$\int_K |\hat{e}_\alpha(\hat{x})|^p \, d\hat{x} > m(K) - \varepsilon$$

where $m(K)$ is the measure of $K$. Therefore

$$\|\hat{e}_\alpha\|_p > (m(K) - \varepsilon)^{1/p} > (m(K)/2)^{1/p}$$

for a small $\varepsilon$, and in general $m(K)$ can be large enough. Hence $\{\|\hat{e}_\alpha\|_p\}$ is not uniformly bounded so is not $\{\|e_\alpha\|^p\}$.

Applying Theorem 1, we can reprove the following result due to Larsen, Liu, and Wang [4; Theorem 5].

**Theorem 2.** *For each $p$ ($1 \leqslant p < \infty$) the following two statements hold*:

(i) *If $I_1$ is a closed ideal in $L^1(G)$, then $I = I_1 \cap A^p(G)$ is a closed ideal in $A^p(G)$.*

(ii) *If $I$ is a closed ideal in $A^p(G)$ and $I_1$ is the closure of $I$ in $L^1(G)$, then $I_1$ is a closed ideal in $L^1(G)$ and $I = I_1 \cap A^p G)$.*

**Remark.** This theorem is also suggested by analogous results in Liu and Wang [5, Theorem 7] in which $A^p(G)$ is replaced by $D = D_{1,p} = L^1(G) \cap L^p(G)$ ($1 < p < \infty$) with the norm $\|f\| = \max(\|f\|_1, \|f\|_p)$.

**Proof of Theorem 2.** The proof of (i) is immediate and will be omitted. Similarly, in (ii) it is easy to verify that $I_1$ is a closed ideal in $L^1(G)$ and that $I \subset I_1 \cap A^p(G)$. We shall prove $I \supset I_1 \cap A^p(G)$ as following.

It sufficies to prove that $I$ is dense in $I_1 \cap A^p(G)$. Let $f \in I_1 \cap A^p(G)$. We show that for any $\varepsilon > 0$ there is an element $h$ in $I$ such that $\|h - f\|^p < \varepsilon$. By Theorem 1, there exists an approximate identity $\{e_\alpha\}$ of $A^p(G)$ for which each $\hat{e}_\alpha$ has compact support in $\hat{G}$. Take a sequence $\{f_n\}$ in $I$ such that $f_n \to f$ in $L^1$-norm. It follows that

$$(1) \qquad e_\alpha * f_n \to e_\alpha * f$$

in $L^1$-norm for each fixed $\alpha$. Now, since

$$\begin{aligned} |\widehat{e_\alpha * f_n}(\hat{x}) - \widehat{e_\alpha * f}(\hat{x})|^p &\leqslant |\hat{e}_\alpha(\hat{x})|^p \, |\hat{f}_n(\hat{x}) - \hat{f}(\hat{x})|^p \\ &\leqslant |\hat{e}_\alpha(\hat{x})|^p \, \|\hat{f}_n - \hat{f}\|_\infty^p \\ &\leqslant |\hat{e}_\alpha(\hat{x})|^p \, \|f_n - f\|_1^p \\ &\leqslant M \, |\hat{e}_\alpha(\hat{x})|^p \end{aligned}$$

for some constant $M$ and $\hat{e}_\alpha \in L^p(\hat{G})$, the Lebesgue convergence theorem is applicable, and

$$(2) \qquad \int_G |\widehat{e_\alpha * f_n}(\hat{x}) - \widehat{e_\alpha * f}(\hat{x})|^p \, d\hat{x} \to 0 \qquad \text{as } n \to \infty$$

for each $\alpha$. Given any $\varepsilon>0$, there exists $\alpha_0$ such that

$$\| e_{\alpha_0}*f-f\|^p<\varepsilon/3$$

and for this $e_{\alpha_0}$ there is an integer $n_0$ such that, by (1) and (2)

$$\| \hat{e}_{\alpha_0}\hat{f}_{n_0}-\hat{e}_{\alpha_0}\hat{f}\|_p<\varepsilon/3$$

and

$$\| e_{\alpha_0}*f_{n_0}-e_{\alpha_0}*f\|_1<\varepsilon/3.$$

Therefore

$$\begin{aligned}\| e_{\alpha_0}*f_{n_0}-f\|^p &\leqslant \| e_{\alpha_0}*f_{n_0}-e_{\alpha_0}*f\|^p+\| e_{\alpha_0}*f-f\|^p\\ &\leqslant \| e_{\alpha_0}*f_{n_0}-e_{\alpha_0}*f\|_1+\| \hat{e}_{\alpha_0}\hat{f}_{n_0}-\hat{e}_{\alpha_0}\hat{f}\|_p+\| e_{\alpha_0}*f-f\|^p\\ &<\varepsilon/3+\varepsilon/3+\varepsilon/3=\varepsilon.\end{aligned}$$

Since $e_{\alpha_0}*f_{n_0}\in I$, this completes the proof. Q.E.D.

**3. Primary ideals in $A^p(G)$.** A primary ideal of a Banach algebra $B$ means a proper ideal $I$ in $B$ that is contained in only one maximal ideal of $B$.

In the Gel'fand representation $\hat{B}$, a primary ideal $I$ can be characterized by the fact that the set on which all functions $\hat{f}(M)\in I$ (where $M$ is a maximal ideal) vanish consists of a single point. If $I$ is a closed primary ideal, then the residue-class algebra $B/I$ contains a unique maximal ideal. The conclusion of Theorem 2 holds also for the case of primary ideals. As the (regular) maximal ideal space of $L^1(G)$ is homeomorphic to the (regular) maximal ideal space of $A^p(G)$, the following proposition holds immediately.

**Proposition 3.** *There is a one-to-one correspondence between the set of all closed primary ideals of $A^p(G)$ and the set of all closed primary ideals of $L^1(G)$. More precisely, every closed primary ideal of $A^p(G)$ is simply the intersection of a unique closed primary ideal of $L^1(G)$ with $A^p(G)$.*

Kaplansky [3] proved that every closed primary ideal in $L^1(G)$ is maximal; and so we have immediately that

**Proposition 4.** *Every closed primary ideal in $A^p(G)$ is maximal; therefore we can identify the set of all closed primary ideals in $A^p(G)$ with $\hat{G}$.*

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