113. *On the Projective Cover of a Factor Module Modulo a Maximal Submodule*[*]

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1. Let R be a ring with 1 which has the Jacobson radical $J(R)$. In [3], Koh has proved the following:

Every irreducible right R-module has a projective cover if and only if R is semiprimary and for any nonzero idempotent $x+J(R)$ in $R/J(R)$ there exists a nonzero idempotent e in R such that $ex-e \in J(R)$.

The purpose of the present paper is, as a generalization of the result of Koh, to show the following theorem:

Theorem. *Let $M=M_R$ be a projective co-atomic module. Then the following statements are equivalent*:

(1) *For every maximal submodule I of M, M/I has a projective cover.*

(2) *$M/J(M)$ is semisimple and for any nonzero idempotent $\hat{s} \in \hat{S}$ there exists nonzero idempotent $e \in S$ such that $\hat{e}\hat{s}=\hat{e}$.*

2. Let $M=M_R$ be an unital right R-module. We write $J(M)$ for the radical of M and $\bar{M}$ for the factor module $M/J(M)$. Let $S=\mathrm{Hom}_R(M, M)$ and let $\hat{S}=\mathrm{Hom}_R(\bar{M}, \bar{M})$. As usual, we write these endomorphisms on the left of their arguments. We note that every $s \in S$ induces an $\hat{s} \in \hat{S}$, since $sJ(M) \subseteq J(M)$. For any submodule U of M, we denote by ν_U the natural epimorphism $M \to M/U$.

A submodule A of M is called *small* if $A+B=M$ for any submodule B of M implies $B=M$. A *projective cover* of M is an epimorphism of a projective module P onto M with small kernel.

We call M is *co-atomic* if every proper submodule of M is contained in a maximal submodule of M. As is easily seen, if M is co-atomic, then $J(M)$ is small in M (cf. [5]). It is well known that M has a maximal submodule if M is projective (cf. [1]), and we can show that semi-perfect modules defined in [4] are co-atomic as follows: Let T be any proper submodule of a semi-perfect module M, and let $P \to M/T \to 0$ be a projective cover of M/T with kernel K. Then $P/K \cong M/T$ and, since any maximal submodule of P contains K, T is contained in a maximal submodule of M as desired.

Lemma 1. *Let M be a projective module and I a maximal*

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submodule of M. *Then* M/I *has a projective cover if and only if there exists a nonzero idempotent* $e \in S$ *such that* eI *is small in* M.

Proof. Let $P \xrightarrow{\pi} M/I \to 0$ be a projective cover of M/I. Since M is projective, there exists a homomorphism $M \xrightarrow{\alpha} P$ such that $\pi\alpha = \nu_I$:

$$\begin{array}{ccccc} & & M & & \\ & {}^{\alpha}\swarrow & \downarrow{\scriptstyle \nu_I} & & \\ P & \xrightarrow{\pi} & M/I & \longrightarrow & 0 \end{array}$$

Then $P = \operatorname{Im}\alpha + \operatorname{Ker}\pi$. Since $\operatorname{Ker}\pi$ is small in P, $P = \operatorname{Im}\alpha$. Since P is projective, the exact sequence $M \xrightarrow{\alpha} P \to 0$ splits, so there exists a homomorphism $P \xrightarrow{\beta} M$ such that $M = \operatorname{Ker}\alpha \oplus \operatorname{Im}\beta$. If $\alpha I = 0$, then $I = \operatorname{Ker}\alpha$. Let $M \xrightarrow{f} I$ be the projection and let $e = 1 - f \in S$. Then we have $eI = (1-f)fM = 0$, ane hence eI is small in M. If $\alpha I \neq 0$, then $\alpha I \subseteq \operatorname{Ker}\pi$ since $\pi\alpha I = \nu_I I = 0$. Now αI is small in P, therefore $\beta\alpha I$ is small in M. Put $e = \beta\alpha \in S$. Then $e^2 = \beta\alpha\beta\alpha = \beta\alpha = e$ and hence e is a desired idempotent.

Conversely, suppose that there is a nonzero idempotent $e \in S$ such that eI is small in M. Put $(I : e) = \{x \in M \mid ex \in I\}$. Since $eI \subseteq J(M)$, the maximality of I implies $(I : e) = I$. Now define a mapping $eM \xrightarrow{g} M/I$ by $g(ex) = x + I$. The mapping g is well defined and is an epimorphism with kernel eI. Since eM is a direct summand of M, eM is projective. Thus g is a projective cover of M/I.

Lemma 2. *Let* I *be a large maximal submodule of* M *and let* $L = \{s \in S \mid sI = 0\}$. *Then* $L^2 = 0$.

Proof. If $s_1 \neq 0$, $s_2 \neq 0$ are elements in L, then $I \cap s_2M \neq 0$. Therefore for some $x \in M$, $0 \neq s_2x \in I$ and $s_1s_2x = 0$. Assume that $s_1s_2 \neq 0$. Then by the maximality of I, $\{y \in M \mid s_1s_2y = 0\} = I$. Thus $x \in I$. This is impossible since $s_2 \in L$ and $s_2x \neq 0$. Thus $L^2 = 0$.

Lemma 3 (cf. [2]). *Let* M *be a co-atomic module such that each maximal submodule of* M *is not large. Then* M *is semisimple.*

Proof. Let F be the socle of M. If $F \neq M$, then F is contained in a maximal submodule I of M. Since I is not large in M, there is a nonzero submodule K of M such that $I \cap K = 0$. By the maximality of I, $M = I \oplus K$. Since $K \cong M/I$, K is irreducible which is not contained in F. This is impossible. Thus $F = M$.

Theorem. *Let* M_R *be a projective co-atomic module. Then the following statements are equivalent*:

(1) *For every maximal submodule* I *of* M, M/I *has a projective cover.*

(2) $M/J(M)$ *is semisimple and for any nonzero idempotent* $\hat{s} \in \hat{S}$ *there exists a nonzero idempotent* $e \in S$ *such that* $\hat{e}\hat{s} = \hat{e}$.

Proof. (1)$\Rightarrow$(2). Since M is co-atomic, $\bar{M}=M/J(M)$ is also co-atomic. Let $\bar{I}$ be a maximal submodule of $\bar{M}$. Then, for some maximal submodule I of M, $\bar{I}=I/J(M)$. By Lemma 1, there exists a nonzero idempotent $e \in S$ such that eI is small in M. Thus $eI \subseteq J(M)$. By the following commutative diagram

$$\begin{array}{ccc} M & \xrightarrow{\quad e \quad} & M \\ {\scriptstyle \nu_{J(M)}} \downarrow & & \downarrow {\scriptstyle \nu_{J(M)}} \\ \bar{M} & \xrightarrow{\quad \hat{e} \quad} & \bar{M} \end{array}$$

$\hat{e}\bar{I}=0$. If $\hat{e}\bar{M}=0$, then $eM \subseteq J(M)$. But, since M is projective, $J(M)$ can not contain any nonzero direct summand of M (cf. [4], p. 350), and hence $\hat{e}$ is a nonzero idempotent in $\hat{S}$. Let $L=\{\hat{s} \in \hat{S} \mid \hat{s}\bar{I}=0\}$. Then $L^2 \neq 0$, and thus, by Lemma 2, $\bar{I}$ is not large in $\bar{M}$. By Lemma 3, $\bar{M}$ is semisimple.

Now let $\hat{s} \in \hat{S}$ be any nonzero idempotent. Since $\bar{M}$ is co-atomic, $(1-\hat{s})\bar{M}$ is contained in a maximal submodule $\bar{I}$ of $\bar{M}$. Then, for some maximal submodule I of M, $\bar{I}=I/J(M)$, and by Lemma 1, there exists a nonzero idempotent $e \in S$ such that eI is small in M. Since $eI \subseteq J(M)$, $\hat{e}\bar{I}=0$. Operating $\hat{e}$ to the relation $(1-\hat{s})\bar{M} \subseteq \bar{I}$, we obtain $\hat{e}(1-\hat{s})\bar{M}=0$. Thus $\hat{e}(1-\hat{s})=0$.

(2)$\Rightarrow$(1). Let I be a maximal submodule of M. Then $J(M) \subseteq I$ and $\bar{I}=I/J(M)$ is a (maximal) submodule of $\bar{M}$. Since $\bar{M}$ is semisimple, there is a (minimal) submodule $\bar{K}=K/J(M)$ such that $\bar{M}=\bar{I} \oplus \bar{K}$. Let $\bar{M} \xrightarrow{\hat{s}} \bar{K}$ be the projection. Then by the assumption, there exists a nonzero idempotent $e \in S$ such that $\hat{e}\hat{s}=\hat{e}$. Since $\hat{s}\bar{I}=0$, $\hat{e}\bar{I}=0$, i.e. $eI \subseteq J(M)$. Now since M is co-atomic, $J(M)$ is small in M, and hence so is eI. By Lemma 1, M/I has a projective cover.

Remark. Since any irreducible right R-module can be written as R/I, where I is a maximal right ideal of R, and since $\hat{S}$ is naturally isomorphic to $S/J(S)$ (cf. [5], p. 95), the above Theorem includes, in the special case where $M_R=R_R$, the result of Koh [3] mentioned in § **1**.

Corollary. *Let M be a projective co-atomic module. Then the following statements are equivalent*:

(1) *M is indecomposable, and for every maximal submodule I of M, M/I has a projective cover.*

(2) *$J(M)$ is the uniqve maximal submodule of M.*

Proof. (1)$\Rightarrow$(2). By the Theorem, $\bar{M}$ is semisimple. Thus for any proper submodule N of M, $\bar{N}=(N+J(M))/J(M)$ is a direct summand of $\bar{M}$. If $\bar{N} \neq 0$, then for the projection $\bar{M} \xrightarrow{\hat{s}} \bar{N}$, there exists a nonzero idempotent $e \in S$ such that $\hat{e}\hat{s}=\hat{e}$. Now since M is indecomposable, $e=1$. Thus $\bar{N}=\hat{s}\bar{M}=\hat{e}\hat{s}\bar{M}=\hat{e}\bar{M}=\bar{M}$. Hence $N+J(M)=M$. Since $J(M)$ is small in M, we obtain $N=M$.

(2)$\Rightarrow$(1). If $M=A+B$, and both A, B are proper submodules of M, then $A\subsetneqq J(M)$ and $B\subsetneqq J(M)$. Thus $M=J(M)$ which is impossible since M is projective. Now since $J(M)$ is small in M, $M\xrightarrow{\nu_{J(M)}}M/J(M)\rightarrow 0$ is a projective cover.

Added in proof. After submitting this paper, Prof. Y. Kurata has proved our Corollary, using Lemma 1, without the assumption of co-atomicness.

References

[1] H. Bass: Finitistic dimension and a homological generalization of semi-primary rings. Trans. Amer. Math. Soc., **95**, 466–488 (1960).

[2] K. Koh: On very large one sided ideals of a ring. Canad. Math. Bull., **9**, 191–196 (1966).

[3] ——: On a semiprimary ring. Proc. Amer. Math. Soc., **19**, 205–208 (1968).

[4] E. A. Mares: Semi-perfect modules. Math. Zeitschr., **82**, 347–360 (1963).

[5] Y. Miyashita: Quasi-projective modules, perfect modules, and a theorem for modular lattices. J. Fac. Hokkaido Univ., **19**, 86–110 (1966).