# 158. On the Bi-ideals in Semigroups

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(Comm. by Kinjirô KUNUGI, M. J. A., Oct. 13, 1969)

Let $S$ be a semigroup, and $A$ be a non-empty subset of $S$. We shall say that $A$ is a *bi-ideal* or *(1, 1)-ideal* of $S$ if the following conditions hold:

(i) $A$ is a subsemigroup of $S$.

(ii) $ASA \subseteq A$.

The notion of bi-ideal was introduced by R. A. Good and D. R. Hughes [2]. It is also a special case of the $(m, n)$-ideal introduced by the author [4].

In this short note we give a summary of some results concerning the bi-ideals of semigroups, and we announce some new results. For the terminology not defined here we refer to the books by A. H. Clifford and G. B. Preston [1]. Proofs of the results will not be given.

**Theorem 1.** *Let $S$ be an arbitrary semigroup. Then any left (right, two-sided, and quasi-) ideal of $S$ is a bi-ideal of $S$.*

**Theorem 2.** *Suppose that $A_1, \cdots, A_n$ are bi-ideals of a semigroup $S$. Then the intersection $B = \bigcap_{i=1}^{n} A_i$ either is empty or it is a bi-ideal of $S$.*

We say that a bi-ideal $A$ of a semigroup $S$ is a *proper bi-ideal* of $S$ if $A$ is a proper subset of $S$, that is, the set $S-A$ is not empty. It is easy to see that a group has not proper bi-ideals, and what is more this property characterizes the class of groups among semigroups.

**Theorem 3.** *A semigroup $S$ is a group if and only if it has not proper bi-ideals.*

By a bi-ideal of a semigroup $S$ generated by a non-empty subset $A$ of $S$ we mean the smallest bi-ideal of $S$ containing $A$. Let us denote this bi-ideal by $(A)_{(1,1)}$. If the set $A$ consists of a single element then the bi-ideal of $S$ generated by $A$ is said to be a *principal bi-ideal* of $S$. It is easy to show that the following assertion is true.

**Theorem 4.** *Let $a$ be an arbitrary element, and $A$ be a non-empty subset of $S$. Then $(A)_{(1,1)} = A \cup A^2 \cup ASA$ and $(a)_{(1,1)} = a \cup a^2 \cup aSa$.*

An important property of the bi-ideals is formulated in the following theorem. This was proved by the author (see [6], first part).

**Theorem 5.** *Let $A$ be a bi-ideal and $B$ be a non-empty subset of $S$. Then the products $AB$ and $BA$ are bi-ideals of $S$.*

Suppose that $\bar{S}$ is the multiplicative semigroup of all non-empty subset of $S$, and $S_1$ is the set of all bi-ideals of $S$. Then by Theorem 5 the set $S_1$ is a semigroup under the multiplication of subsets, and the subsemigroup $S_1$ is a two-sided ideal of $\bar{S}$.

**Theorem 6.** *Let $A$, $B$ be bi-ideals of the semigroup $S$. Then the products $AB$ and $BA$ are also bi-ideals of $S$.*

As a simple consequence of Theorem 6 we obtain the following result.

**Theorem 7.** *Let $P$, $Q$ be quasi-ideals of a semigroup $S$. Then the products $PQ$ and $QP$ are bi-ideals of $S$.*

It is known the following characterizations of the bi-ideal. (See [5] and [1].)

**Theorem 8.** *A non-empty subset $B$ of a semigroup $S$ is a bi-ideal of $S$ if and only if any one of the following assertions holds:*

(A) *There exists a left ideal $L$ of $S$ such that $B$ is a right ideal of $L$.*

(B) *There exists a right ideal $R$ of $S$ so that $B$ is a left ideal of $R$.*

(C) *There exists a left ideal $L$ and a right ideal $R$ of $S$ such that*

$$RL \subseteq B \subseteq R \cap L. \tag{1}$$

In what follows we shall say that $S$ is *regular* if to any element $a$ of $S$ there exists an element $x$ in $S$ such that the condition

$$axa = a \tag{2}$$

holds. It is known that a semigroup $S$ is regular if and only if the relation

$$L \cap R = RL \tag{3}$$

holds for any left ideal $L$ and for any right ideal $R$ of $S$. This criterion and Theorem 8 imply the following result.

**Theorem 9.** *Let $S$ be a regular semigroup and $A$ be a non-empty subset of $S$. Then $A$ is a bi-ideal of $S$ if and only if it may be represented in the form*

$$A = RL, \tag{4}$$

*where $L$ is a left ideal and $R$ is a right ideal of $S$.*

The author recently obtained the following characterizations of regular semigroups by means of bi-ideals.

**Theorem 10.** *A semigroup $S$ is regular if and only if*

$$(a)_{(1,1)} = aSa \tag{5}$$

*for each element $a$ of $S$.*

**Theorem 11.** *A semigroup $S$ is regular if and only if*

$$(a)_{(1,1)} = (a)_R (a)_L \tag{6}$$

*for every element $a$ of $S$. $(a)_L$ denotes the principal left ideal of $S$ generated by the element $a$ in $S$.*

A semigroup $S$ is said to be a *duo semigroup* if every one-sided (left or right) ideal of $S$ is a two-sided ideal. Theorem 9 has an interesting consequence for the case of regular duo semigroups.

**Theorem 12.** *Let S be a regular duo semigroup. Then every bi-ideal of S is a two-sided ideal of S.*

It is known that a semigroup $S$ which is a semilattice of groups is both regular and duo semigroup. (See the author's paper [10].) Therefore Theorem 12 implies the following result.

**Theorem 13.** *Let S be a semigroup which is a semilattice of groups. Then each bi-ideal of S is a two-sided ideal of S.*

**Corollary.** *Let S be a semigroup which is a semilattice of groups. Then every quasi-ideal Q of S is a two-sided ideal of S.*

It may be noted that Theorems 12, 13 remain true with $(m, n)$-ideal instead of bi-ideal.

**Theorem 14.** *Suppose that S is a regular duo semigroup, and* $m, n$ *are arbitrary non-negative integers such that* $m+n>0$. *Then every* $(m, n)$*-ideal of S is a two-sided ideal of S.*

**Theorem 15.** *Let S be a semigroup which is a semilattice of groups, and let* $m, n$ *are arbitrary non-negative integers such that* $m+n>0$. *Then any* $(m, n)$*-ideal of S is a two-sided ideal of S.*

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