155. *Representation of Certain Banach $*$-algebras**)

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(Comm. by Kinjirô KUNUGI, M. J. A., Oct. 13, 1969)

Let A be a Banach $*$-algebra satisfying the condition: there exists a positive constant α such that

$$\alpha\|x^*\|\|x\|\leq\|x^*x\|$$

for every x in A. The problem to realize such a Banach $*$-algebra as a C^*-algebra has been left to be solved after I. Kaplansky [3] asked whether or not every C-symmetric Banach $*$-algebra is symmetric. In the case when A is commutative, R. Arens [1] had proven that it is a B^*-algebra under an equivalent norm, and then B. Yood [8] gave a partial answer to this problem by showing that a Banach $*$-algebra with the above condition is a B^*-algebra under an equivalent norm if $\alpha>c$ (c; the unique real root of the equation $4t^3-2t^2+t-1=0$).

The purpose of this note is to inform that this problem has been solved in the affirmative, and is to give a brief account of the proof. Our result is the following.

Theorem. *Let A be a Banach $*$-algebra whose norm satisfies the condition $\alpha\|x^*\|\|x\|\leq\|x^*x\|$. Then it is homeomorphic and $*$-isomorphic to a C^*-algebra.*

By a B^*-algebra, we shall mean a Banach $*$-algebra with the condition $\|x^*x\|=\|x\|^2$. At the present time, it is well known that a B^*-algebra is isometrically $*$-isomorphic to a C^*-algebra, a uniformly closed $*$-algebra of operators on Hilbert space.

Throughout this paper we shall consider a (complex) Banach $*$-algebra with unit e (the case without unit will be mentioned at the final step). Here we present a concise proof of the theorem which proceeds by stages. In the course of the representation of B^*-algebras (see the theorem of Fukamiya and Kaplansky [7; Theorem 4.8. 11], T. Ono [6] and J. Glimm-R. V. Kadison [2]), the problem one discussed for a long time was to extend the local C^*-property to the global one. Concerning our problem we are in the same situation as the case of B^*-algebras because Arens [1] tells us that our Banach $*$-algebras provide the local C^*-property. To clarify the essence of the proof we introduce a class of Banach $*$-algebras as follows. A Banach $*$-algebra A is said to be *locally equivalent to a C^*-algebra* if every maximal

*) This work was done while the author was offering a Functional Analysis Seminar at the University of British Columbia in 1969.

commutative $*$-subalgebra of A is a B^*-algebra under an equivalent norm. For such a Banach $*$-algebra A, first of all, we should observe the following facts.

(i) The (complex) closed subalgebra $A(h)$ of A generated by a self-adjoint element h and e is a B^*-algebra under an equivalent norm, that is, it is homeomorphic and $*$-isomorphic to the algebra of all continuous functions on a compact Hausdorff space.

(ii) The spectrum of h in A coincides with that of h in $A(h)$. Consequently, we have

(iii) Every self-adjoint element in A has real spectrum.

If a Banach $*$-algebra A satisfies the norm condition stated in the theorem, then it follows from [1; Theorem 1] that A is locally equivalent to a C^*-algebra in our sense (note that every maximal commutative $*$-subalgebra of A is automatically closed).

In what follows, A means a Banach $*$-algebra which is locally equivalent to a C^*-algebra. We denote by $\sigma(x)$ the spectrum of an element x; $\gamma(x)$ the spectral radius of x; S the set of all self-adjoint elements in A. An element $h \in S$ is said to be positive if $\sigma(h)$ consists of non-negative scalars, and is denoted by $h \geq 0$. By a state ρ of A, we understand a linear functional of A such that $\rho(h) \geq 0$ for every $h \geq 0$ and $\rho(e)=1$. The first step is to prove:

(1) *The following conditions are equivalent in A.*

(1.1) *The sum of positive elements is positive.*

(1.2) *For each element x, x^*x is positive.*

(1.3) *For any $h, k \in S$, $\gamma(h+k) \leq \gamma(h)+\gamma(k)$.*

(1.4) *Let $h \in S$ and $\lambda \in \sigma(h)$. Then there exists a state ρ of A such that $\rho(h)=\lambda$.*

From this it turns out that a key to the problem is to prove that one of the conditions (1.1)–(1.4) holds for A. Actually we can prove:

(2) A satisfies the condition (1.1).

By (i), an element in A is positive if and only if it is expressible as the square of an element in S. Therefore, to prove (2) we may show that for a, b in S, a^2+b^2 is positive; that is, $a^2+b^2+\lambda e$ is invertible for any $\lambda > 0$. Following the procedure as in [5; IX, p. 302] this can be reduced to show that for any invertible element x, x^*x is positive. Consider the algebra $A_0 = A(x^*x)$ generated by x^*x and e. Then A_0 is $*$-isomorphic to the algebra $C(\Omega)$ of all continuous functions on the compact Hausdorff space Ω (the space of all multiplicative linear functionals φ on A_0 with $\varphi(e)=1$). Having noticed that x^*x has the inverse in A_0, we define

$$\Omega_1 = \{\varphi \in \Omega \mid (x^*x)^\wedge(\varphi) > 0\};$$
$$\Omega_2 = \{\varphi \in \Omega \mid (x^*x)^\wedge(\varphi) < 0\},$$

where $(x^*x)^\wedge$ is the image of x^*x under the $*$-isomorphism of A_0 onto $C(\Omega)$. Then $\Omega=\Omega_1\cup\Omega_2$ and further Ω_1, Ω_2 are open and closed in Ω. Let us denote by p the element in A_0 corresponding to the characteristic function of Ω_2. Obviously p is a projection (self-adjoint and idempotent) commuting with x^*x. Here $x^*x(e-p)\geq 0$ and $x^*xp\leq 0$. Thus the proof will complete by showing $p=0$. Suppose $p\neq 0$, i.e., $\Omega_2\neq\varnothing$. Then, since Ω_2 is compact, there is a $\varphi_0\in\Omega_2$ such that

$$\sup_{\varphi\in\Omega_2}(px^*xp)^\wedge(\varphi)=(px^*xp)^\wedge(\varphi_0)<0.$$

This means that there is a constant $\beta<0$ such that $px^*xp\leq\beta p$, so that we may assume without loss of generality that $px^*xp\leq -p$ holds. Note that $p(x^*x)^{-1}p=(x^*x)^{-1}p\leq 0$ and find a self-adjoint element h in A_0 such that $(x^*x)^{-1}p=-h^2$. Let $y=xh$. Then $yp=y$ and $y^*y=-p$. Define $z=y-py$ and observe that $z^2=0$. Thus it follows from (i) that $z^*z\geq 0$ (see [5; Lemma 1.1]). On the other hand, a direct computation shows $z^*z=-p-y^*py$ and hence $(py)^*(py)=y^*py\leq -p$. Here we should notice that $w=py$ commutes with p. Clearly the set B of all elements which commute with p forms a closed $*$-subalgebra of A, and p belongs to the center of B. Therefore, restricting our attention to pBp, it is possible to assume that $w^*w\leq -e$. In this case, since $(w^*w)^{-1}$ exists and it is negative, there is an element $k\in S$ commuting with w^*w such that $(w^*w)^{-1}=-k^2$. Let $u=wk$. Then we have

$$u^*u=-e,\ uu^*=-q(q;\text{ a projection}).$$

Suppose $q'=e-q\neq 0$. Put $v=uq'$ and observe that $q'u=0$. Then $q'v=0$, $vq'=v$ and $v^2=0$. In this case, as mentioned above, $v^*v\geq 0$. But $v^*v=q'u^*uq'=-q'<0$, which is a contradiction.[1] Thus $q=e$ and so u is normal. Therefore $u^*u=-e$ is impossible by our assumption. This completes the proof of (2).

After having established the proposition (2), there is no difficulty to obtain a faithful $*$-representation of A into the algebra of operators on Hilbert space. What we need is only to follow the standard treatment in the representation of Banach $*$-algebras (cf. [6]).

(3.1) *S is closed in A.*

(3.2) *The involution *is continuous in A.*

(3.3) *A state of A is bounded.*

Now, according to (1), (2) and (3), we construct a $*$-representation π_ρ of A on a Hilbert space π_ρ, associated with each state ρ of A, and consider the direct sum π of π_ρ. Then, by (1.4), π is faithful.

(4) *A admits a faithful $*$-representation into the algebra $\mathcal{L}(H)$*

1) This argument has been inspired by [3; Lemma 4.2, (c)]. Indeed, q′ is minimal in the closed subalgebra generated by polynomials $p(u,u^*)$. The structure The structure theorem of isometries on Hilbert space suggests us this fact. See also [4; Lemma 4].

of all operators on a Hilbert space H.

Let us return to a Banach $*$-algebra A in Theorem. By (4), it is $*$-isomorphic to a $*$-subalgebra $\pi(A)$ of $\mathcal{L}(H)$, and furthermore the norm condition yields $\alpha\|h\|\leq\gamma(h)$ for every $h\in S$([1]). Therefore $\alpha\|h\|\leq\|\pi(h)\|\leq\|h\|$ for every $h\in S$. This implies that $\pi(S)$ is closed in $\mathcal{L}(H)$ and hence $\pi(A)$ is closed in $\mathcal{L}(H)$. That is, $\pi(A)$ is a C^*-algebra and π is a homeomorphism.

In the case when a Banach $*$-algebra in Theorem has no unit, it can be isometrically embedded into a Banach $*$-algebra with unit which satisfies the norm condition as in the theorem ([1; Lemma 4]).

Acknowledgements. Professor B. Yood has informed me of the following fact: S. Shirai and J.W.M. Ford recently gave an affirmative answer to the problem of I. Kaplansky stated in the introduction. Then, their result contains the statement (2) and so our theorem also follows from this with the aid of (1), (3) and (4).

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