149. A Remark on a Semilinear Degenerate Diffusion System

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(Comm. by Kinjirô KUNUGI, M. J. A., Oct. 13, 1969)

§1. **Introduction.** This remark is concerned with the following mixed problem in $R^T=\{0<t\leqslant T,\ 0<x\}$,

$$\frac{\partial u}{\partial t}=\frac{\partial^2 u}{\partial x^2}+f(v)u+g(v), \qquad \frac{\partial v}{\partial t}=u, \tag{1}$$

with the initial boundary conditions,

$$\begin{array}{lll} u(x,0)=u_0(x), & v(x,0)=v_0(x) & \text{for } 0\leqslant x \\ v(0,t)=\psi(t) & \text{for } 0\leqslant t\leqslant T. & \end{array} \tag{2}$$

First, let us note the theorem proved by R. Arima and Y. Hasegawa [1] with respect to the problem (1) and (2), which is given as follow:

Theorem 1. *Suppose,*

$$\begin{array}{ll} f(v),\ g(v)\in C^1, & \\ -K_1(v^2+1)\leqslant f(v)\leqslant L, & \\ |g(v)|\leqslant K_2(v^2+|v|) \quad \text{and} \quad G(v)\equiv\int_0^v g(z)dz\leqslant K_3v^2, & \\ u_0(x),\quad v_0(x)\in \mathcal{B}^2_+\cap\mathcal{D}^2_{L^2+} & \text{for } 0\leqslant x, \\ \psi(t)\in C^2 & \text{for } 0\leqslant t\leqslant T, \\ u_0(0)=\varphi'(0), \quad v_0(0)=\varphi(0), & \\ \psi''(0)=u_0''(0)+f(\psi(0))\psi'(0)+g(\psi(0)). & \end{array} \tag{3}$$

Then there exists a unique solution $\{u(x,t),v(x,t)\}$ *in* R^T *such that* $\{u(x,t),v(x,t)\}\in\mathcal{E}^0_t(\mathcal{B}^2_+\cap\mathcal{D}^2_{L^2+})$, *where* L, K_1, K_2, *and* K_3 *are positive constants.*

In this note we prove the existence and the uniqueness theorem of the following more general system than (1) by using a suitable difference scheme,

$$\begin{aligned} \frac{\partial u}{\partial t}&=\frac{\partial^2 u}{\partial x^2}+f(v)u+g(v) \\ \frac{\partial v}{\partial t}&=a(u)v+b(u) \end{aligned} \tag{4}$$

and drive the different conditions from (3) in the case of $a(u)\equiv 0$ and $b(u)\equiv u$.

Here we consider the mixed problem in R^T for (4) with the initial boundary conditions,

$$\begin{array}{lll} u(x,0)=u_0(x), & v(x,0)=v_0(x) & \text{for } 0\leqslant x \\ u(0,t)=\varphi(t), & v(0,t)=\psi(t) & \text{for } 0\leqslant t\leqslant T, \end{array} \tag{5}$$

and also the compatibility conditions,

$$(6)\quad \begin{aligned} &\varphi'(0)=u_0''(0)+f(\psi(0))\varphi(0)+g(\psi(0)),\\ &u_0(0)=\varphi(0), \qquad v_0(0)=\psi(0),\\ &\psi'(t)=a(\varphi(t))\psi(t)+b(\varphi(t)) \qquad \text{for } 0\leqslant t\leqslant T. \end{aligned}$$

§ 2. Existence theorem.

Let us introduce a difference scheme to (4):

$$(7)\quad \begin{aligned} &\frac{u^{n+1,j}-u^{n,j}}{k}=\frac{u^{n,j+1}-2u^{n,j}+u^{n,j+1}}{h^2}+f(v^{n,j})u^{n+1,j}+g(v^{n,j}),\\ &\frac{v^{n+1,j}-v^{n,j}}{k}= \qquad a(u^{n,j})v^{n+1,j}+b(u^{n,j}). \end{aligned}$$

We consider (7) for $j=1, 2, \cdots$ and $n=0, 1, \cdots, N$ with the initial boundary conditions,

$$(8)\quad \begin{aligned} &u^{0,j}=u_0(jh), \qquad v^{0,j}=v_0(jh) \qquad \text{for } j=0, 1, 2, \cdots,\\ &u^{n,0}=\varphi(nk), \qquad v^{n,0}=\psi(nk) \qquad \text{for } n=0, 1, 2, \cdots, N, \end{aligned}$$

and the compatibility conditions

$$(9)\quad \begin{aligned} &\frac{u^{1,0}-u^{0,0}}{k}=\frac{u^{0,1}-2u^{0,0}+u^{0,-1}}{h^2}+f(v^{0,0})u^{1,0}+g(v^{0,0}),\\ &\frac{v^{n+1,0}-v^{n,0}}{k}=a(u^{n,0})v^{n-1,1}+b(u^{n,0}) \qquad \text{for } n=0, 1, 2, \cdots, N. \end{aligned}$$

Here $w^{n,j}=w(jh, nk)$ for $w\equiv u$ or v and n, k, $N=\dfrac{T}{k}-1$ are integers.

Now we have the following lemma.

Lemma 1. *Supposing the conditions;*

$$(10)\quad \begin{aligned} &\frac{k}{h^2}\leqslant\frac{1}{2}, \qquad k<\frac{1}{L},\\ &f(v),\ a(u)\leqslant L,\\ &|g(v)|\leqslant M_1|v|, \qquad |b(u)|\leqslant M_2|u|, \end{aligned}$$

where L, M_1 and M_2 are positive constants, then the solution of the difference scheme (7) *under the initial boundary conditions* (8) *is stable.*

The proof is the following. (7) is written as follows,

$$(11)\quad \begin{aligned} &u^{n+1,j}=\frac{1}{1-kf(v^{n,j})}\{P(u^{n,j})+kg(v^{n,j})\},\\ &v^{n+1,j}=\frac{1}{1-ka(u^{n,j})}\{v^{n,j}+kb(u^{n,j})\}, \end{aligned}$$

where $P(u^{n,j})=\dfrac{ku^{n,j+1}+(1-2k)u^{n,j}+ku^{n,j-1}}{h^2}$. From (8), (10) and (11),

$$(12)\quad \begin{aligned} &\max(|u^{n+1}|, |\varphi|)\leqslant\frac{1}{1-kL}\{\max(|u^n|, |\varphi|)+\max(|v^n|, |\psi|)M_1k\},\\ &\max(|v^{n+1}|, |\psi|)\leqslant\frac{1}{1-kL}\{\max(|v^n|, |\psi|)+\max(|u^n|, |\varphi|)M_2k\}, \end{aligned}$$

where $|w^n|=\sup\limits_{j\geq 0}|w^{n,j}|$, $|\chi|=\sup\limits_{N+1\geq n\geq 0}|\chi^n|$ for $\chi\equiv\varphi$ or ψ. Thus the following estimates are obtained for any n,

$$H^{n+1} \leqslant \frac{1+kM}{1-kL} \cdot H^n$$

and also

$$H^n \leqslant e^{(M+L)T} \cdot H^0 \qquad \text{for } n=1, 2, \cdots, N+1,$$

where $H^n = \max(|u^n|, |\varphi|) + \max(|v^n|, |\psi|)$ and $M = \max(M_1, M_2)$. Lemma is proved.

Proposition. *Supposing the conditions;*

$$(13) \quad \begin{aligned} & u_0(x) \in \mathcal{B}^4_+, \quad v_0(x) \in \mathcal{B}^1_+, \\ & \varphi(t) \in C^2, \\ & f(v), g(v) \in C^2 \text{ and } a(u), b(u) \in C^1, \\ & f(v), a(u) \leqslant L \text{ and } |g(v)| \leqslant M_1 |v|, \quad |b(u)| \leqslant M_2 |u|, \end{aligned}$$

then there exists the genuine solution of the problem (4), (5) *and* (6) *in* R^T *such that*

$$(4) \quad u(x, t) \in \mathcal{E}^0_t(\mathcal{B}^2_+) \cap \mathcal{E}^1_t(\mathcal{B}^1_+), \qquad v(x, t) \in \mathcal{E}^1_t(\mathcal{B}^0_+).$$

Proposition is proved by a slight modification of the argument [2]. If higher derivatives of u_0, v_0 and φ are bounded, it is possible to select a subsequence of the h, for which $\{u^{n,j}, v^{n,j}\}$ converges together with a number of its derivatives by using the help of Lemma 1 and the limit function $\{u(x, t), v(x, t)\}$ is a solution of (4), (5) and (6). Here the proof is omitted.

Theorem 2. *Supposing the conditions;*

$$(15) \quad \begin{aligned} & u_0(x) \in \mathcal{B}^2_+, \quad v_0(x) \in \mathcal{B}^1_+ \\ & \varphi(t) \in C^1 \text{ or } \psi(t) \in C^2 \\ & f(v), g(u), a(v), b(u) \in C^1 \\ & f(v), a(u) \leqslant L \text{ and } |g(v)| \leqslant M_1 |v|, \quad |b(u)| \leqslant M_2 |u|, \end{aligned}$$

then there exists the genuine solution of the problem (4), (5) *and* (6) *in* R^T *such that*

$$(14) \quad u(x, t) \in \mathcal{E}^0_t(\mathcal{B}^2_+) \cap \mathcal{E}^1_t(\mathcal{B}^1_+), \qquad v(x, t) \in \mathcal{E}^1_t(\mathcal{B}^0_+).$$

Theorem 2 can be proved by using the properties of the fundamental solution of heat equation.

§3. Uniqueness theorem.

We have the following lemma.

Lemma 2. *If* $\{u(x, t), v(x, t)\}$ *is a solution of the problem* (4), (5) *and* (6) *in* R^T, *for which* $u(x, t) \in \mathcal{E}^0_t(\mathcal{B}^2_+) \cap \mathcal{E}^1_t(\mathcal{B}^0_+)$ *and* $v(x, t) \in \mathcal{E}^1_t(\mathcal{B}^0_+)$ *and if* $\{u^{n,j}, v^{n,j}\}$ *is the solution of the problem* (7), (8) *and* (9) *under the conditions* (10), *then there exists a* $\delta(\varepsilon)$ *for any* ε, *such that for* $0 < h, k \leqslant \delta$,

$$(16) \quad \|u^{n,j} - u(x, t)\| + \|v^{n,j} - v(x, t)\| < \varepsilon \quad \text{in } R^T_h,$$

where $\|w\| = \sup_{R^T_h} |w(x, t)|$ *and* $R^T_h = \{$*the rectangular lattices with mesh sizes* (h, k) *in* $R^T\}$.

The proof is analogous to that of [2]. So it is omitted.

Theorem 3. *As for a genuine solution* $\{u(x, t), v(x, t)\}$ *of* (4), (5) *and* (6) *satisfying the assumption of Lemma* 2, *the solution is unique.*

The proof is that, if $\{u_1(x, t), v_1(x, t)\}$ and $\{u_2(x, t), v_2(x, t)\}$ are both arbitrary functions satisfying (16) of Lemma 2, then for $0<k, h\leqslant\delta$,

$$\|u_1(x, t)-u_2(x, t)\|+\|v_1(x, t)-v_2(x, t)\|\leqslant\|u_1(x, t)-u^{n,j}\|$$
$$+\|u_2(x, t)-u^{n,j}\|+\|v_1(x, t)-v^{n,j}\|+\|v_2(x, t)-v^{n,j}\|<2\varepsilon \quad \text{in} \quad R_h^T,$$

where $\{u^{n,j}, v^{n,j}\}$ is the solution of (7), (8) and (9). Thus we can prove Theorem 3.

Acknowledgment. The author would like to thank Professor K. Okugawa and Professor M. Yamaguti for valuable suggestions and many criticisms.

References

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