148. *Korteweg-deVries Equation.* IV

Simplest Generalization

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1. Introduction. In the preceding note [1], [2] we have established the global existence theorem for the Cauchy problem for the KdV equation

$$(1)\qquad \begin{cases} D_t u + uDu + D^3 u = 0 & (x,t) \in R^1 \times [0,\infty) \\ u(x,0) = f(x) & x \in R^1 \quad \left(D_t = \dfrac{\partial}{\partial t},\ D = \dfrac{\partial}{\partial x}\right). \end{cases}$$

Here we consider its simplest generalization

$$(2)\qquad \begin{cases} D_t v + v^2 Dv + D^3 v = 0 & (x,t) \in R^1 \times [0,\infty) \\ v(x,0) = f(x) & x \in R^1. \end{cases}$$

These Cauchy problems are closely related to the study of anharmonic lattices [3]. Recently Miura [4] has discovered a remarkable nonlinear transformation

$$(3)\qquad \sqrt{-6}Dv + v^2 = u$$

which connects (1) and (2) in the following manner

$$(4)\qquad D_t u + uDu + D^3 u = (2v + \sqrt{-6}D)(D_t v + v^2 Dv + D^3 v).$$

For any smooth solution $v(\in \mathcal{E}_t^\infty(\mathcal{E}_{L^2}^\infty))$ of (2) we can associate uniquely the solution $u(\in \mathcal{E}_t^\infty(\mathcal{E}_{L^2}^\infty))$ of (1) through the transformation (3). But converse is not true. When we want to solve the equation (3) with respect to v for given u we have no uniqueness. First we show the example which violates the uniqueness of the solutions of the equation (3). Let $\varphi(x) \in \mathcal{E}_{L^2}^\infty$ be such that $\varphi(x) > 0$ for $^\forall x \in R^1$, $\varphi(x) = \dfrac{1}{|x|}$ for $|x| > R$ for some $R > 0$. We define v, w, u as follows

$$v = \frac{1}{2}\varphi - \frac{\sqrt{-6}}{2}\frac{D\varphi}{\varphi}, \qquad w = v - \varphi, \qquad u = \sqrt{-6}Dv + v^2.$$

Then v and w are distinct each other and satisfies the same equation (3). That is the violence of the uniqueness of the equation (3). Therefore the global existence theorem for the Cauchy problem (1) is insufficient for the global existence theorem for the Cauchy problem (2). We establish here the global existence theorem for the Cauchy problem (2) in a slightly general situation. Detailed proof will be published elsewhere.

2. Main theorem.

Consider the Cauchy problem for the generalized KdV equation.

$$(5)\quad \begin{cases} D_t v + v^2 Dv + D^3 v + a(x,t)Dv + b(x,t)v + g(x,t) = 0 \\ \qquad\qquad\qquad\qquad\qquad\qquad (x,t) \in R^1 \times [0,\infty) \\ v(x,0) = f(x) \qquad\qquad\qquad x \in R^1. \end{cases}$$

Here we assume $a(x,t), b(x,t) \in \mathcal{E}_t^\infty(\mathcal{B}^\infty)$

Main theorem.

If

$$f(x) \in \mathcal{E}_{L^2}^{3(k+1)+2}$$
$$g(x,t) \in \mathcal{E}_t^{k+1}(\mathcal{E}_{L^2}^2) \cap [\mathcal{E}_t^k(\mathcal{E}_{L^2}^2) \cap \mathcal{E}_t^{k-1}(\mathcal{E}_{L^2}^5) \cap \cdots \cap \mathcal{E}_t^0(\mathcal{E}_{L^2}^{3k+2})]$$

then there exist uniquely the global solution

$$v(x,t) \in \mathcal{E}_t^k(\mathcal{E}_{L^2}^2) \cap \mathcal{E}_t^{k-1}(\mathcal{E}_{L^2}^5) \cap \cdots \cap \mathcal{E}_t^0(\mathcal{E}_{L^2}^{3k+2})$$

of the Cauchy problem (5).

Using Sobolev's lemma we obtain

Corollary 1.

If

$$f(x) \in \mathcal{E}_{L^2}^{3(k+2)+2}$$
$$g(x,t) \in \mathcal{E}_t^{k+2}(\mathcal{E}_{L^2}^2) \cap [\mathcal{E}_t^{k+1}(\mathcal{E}_{L^2}^2) \cap \mathcal{E}_t^k(\mathcal{E}_{L^2}^5) \cap \cdots \cap \mathcal{E}_t^0(\mathcal{E}_{L^2}^{3(k+1)+2})$$

then for any i, j such that $i+j \le k$

$$D_t^i D^{3j} v(x,t) \in \mathcal{B}^0(R^1 \times [0,T]) \qquad \text{for } {}^\forall T > 0.$$

Remark. In Corollary 1 if we take $k=1$ we obtain the global existence theorem of the classical solution.

Corollary 2.

If

$$f(x) \in \mathcal{E}_{L^2}^\infty, \qquad g(x,t) \in \mathcal{E}_t^\infty(\mathcal{E}_{L^2}^\infty)$$

then

$$v(x,t) \in \mathcal{E}_t^\infty(\mathcal{E}_{L^2}^\infty)$$

especially

$$v(x,t) \in \mathcal{B}^\infty(R^1 \times [0,T]) \qquad \textit{for } {}^\forall T > 0.$$

Remark. These results are also true when the functional space $\mathcal{E}_{L^2}^k$ are replaced by the functional space $\mathcal{P}_l^k$ for any $l > 0$. In this case we must assume that $a(x,t)$ and $b(x,t)$ have period l in x. Here $\mathcal{P}_l^k$ represents the functional space which consists of all the functions having the period l and belonging to the functional space $\mathcal{E}_{L^2_{\mathrm{loc}}}^k(R^1)$.

3. Proof of the main theorem. To prove the main theorem we need global a priori estimate and local existence theorem. To obtain a priori estimate of the solutions of the Cauchy problem (5) we use the infinite sequence of polynomial conserved densities (definition will be found in [1] or [5]) of the generalized KdV equation (2) which is obtained from that of the KdV equation (1) replacing u by the left hand member of the equation (3). We can assert the existence of infinite sequence of polynomial conserved densities of the generalized

KdV equation in the following canonical form.

Theorem 1. *The generalized KdV equation* (2) *has the polynomial conserved densities of the form*

$$S_0(v)=v^2$$
$$S_1(v)=(Dv)^2-\frac{1}{6}v^4$$
$$S_2(v)=(D^2v)^2-\frac{5}{3}v^2(Dv)^2-\frac{1}{\sqrt{-6}}v^4Dv+\frac{1}{18}v^6$$
$$S_k(v)=(D^kv)^2+P_k(v, Dv)(D^{k-1}v)^2+Q_k(v, \cdots, D^{k-2}v)D^{k-1}v$$
$$+R_k(v, \cdots, D^{k-2}v) \qquad k=3, 4, \cdots.$$

Here P_k, Q_k and R_k are polynomials.

Integrating these conserved densities on the whole x-axis we get step by step following infinite sequence of a priori estimate of the solutions of the Cauchy problem (5).

Theorem 2. *The solutions of the Cauchy problem* (5) *satisfy following infinite sequence of a priori estimate*

$$\|D^kv\|\leq V_k(t, |a|_t, \cdots, |D^ka|_t, |b|_t, \cdots, |D^kb|_t,$$
$$\|f\|, \cdots, \|D^kf\|, |||g|||_t, \cdots, |||D^kg|||_t) \qquad k=0, 1, 2, \cdots.$$

Here V_k are positive valued smooth monotone increasing function in each arguments. V_0 contains $|Da|_t$ exceptionally.

$$|a|_t=\sup_{0\leq s\leq t}|a(s)|, \qquad |a|=|a|_{\mathscr{B}^0},$$
$$|||g|||_t=\sup_{0\leq s\leq t}\|g(s)\|, \qquad \|g\|=\|g\|_{L^2}$$

The local existence theorem is obtained by the method of successive approximation

$$(6) \qquad v_0(x, t)=f(x) \qquad (x, t)\in R^1\times[0, \infty)$$

$$(7) \qquad \begin{cases} D_tv_n+v_{n-1}^2Dv_n+D^3v_n+a(x, t)Dv_n+b(x, t)v_n+g(x, t)=0 \\ \qquad\qquad\qquad\qquad\qquad\qquad (x, t)\in R^1\times[0, \infty) \\ v_n(x, 0)=f(x) \qquad\qquad\qquad\qquad x\in R^1 \end{cases}$$

By induction in n and k we obtain the uniform (with respect to n) energy estimate for the sequence of approximate solutions v_n.

Proposition 1. *For any non negative integer k, if we take*

$$t_k=\min\left\{1, \frac{\log M}{C_k}\right\} \ (M>1 \textit{ fixed})$$

then we have

$$\sup_{0\leq t\leq t_k}\|D_t^kv_n\|\leq c_k, \qquad \sup_{0\leq t\leq t_k}\|D_t^iD^{3j}v_n\|\leq C_k \qquad (i+j\leq k).$$

Here we use the following notations

$$v(0)=f(x)$$
$$v^{(1)}(0)=-[f^2(x)Df(x)+D^3f(x)+a(x, 0)Df(x)$$
$$+b(x, 0)f(x)+g(x, 0)]$$

$$v^{(k)}(0)=-\Big[\sum_{\alpha_1+\alpha_2+\alpha_3=k-1}\frac{(k-1)!}{\alpha_1!\alpha_2!\alpha_3!}v^{(\alpha_1)}(0)v^{(\alpha_2)}(0)Dv^{(\alpha_3)}(0)$$
$$+D^3v^{(k-1)}(0)+\sum_{\beta=0}^{k-1}\binom{k-1}{\beta}D_t^{k-1-\beta}a(x,0)Dv^{(\beta)}(0)$$
$$+\sum_{\beta=0}^{k-1}\binom{k-1}{\beta}D_t^{k-1-\beta}b(x,0)v^{(\beta)}(0)+D_t^{k-1}g(x,0)\Big]$$
$$c_k^2=M\Big[\|v^{(k)}(0)\|^2_{\mathcal{E}^2_{L^2}}+2\sum_{l=0}^{k-1}c_l^2+\sup_{0\le t\le 1}\|D_t^k g\|^2_{\mathcal{E}^2_{L^2}}\Big]$$

C_k is a polynomial in $c_0,\cdots,c_k$, $\sup_{0\le t\le 1}|D_t^iD^{3j}a|_{\mathcal{B}^2}$, $\sup_{0\le t\le 1}|D_t^iD^{3j}b|_{\mathcal{B}^2}$ $(i+j\le k)$, and $\sup_{0\le t\le 1}\|D_t^iD^{3j}g\|_{\mathcal{E}^2_{L^2}}$ $(i+j\le k-1)$ with positive coefficients.

From (7) we derive the following equation for the differences $v_{n+1}-v_n=\varphi_n$

$$\begin{cases}D_t\varphi_n+v_n^2D\varphi_n+D^3\varphi_n+a(x,t)D\varphi_n+b(x,t)\varphi_n\\ \quad+\varphi_{n-1}(v_n+v_{n-1})Dv_n=0\\ \varphi_n(x,0)=0\end{cases}$$

Using uniform estimate for v_n in Proposition 1 we obtain the following estimate for φ_n.

Proposition 2. *For any non negative integer k, if take*

$$T_k=\frac{\rho}{C_{k+1}}\qquad (0<\rho<1 \text{ fixed})$$

then we have

$$\sup_{0\le t\le T_k}\sum_{i=0}^{k}\|D_t^i\varphi_n\|^2_{\mathcal{E}^2_{L^2}}\le\rho\sup_{0\le t\le T_k}\sum_{i=0}^{k}\|D_t^i\varphi_{n-1}\|^2_{\mathcal{E}^2_{L^2}}.$$

From this estimate it follows easily that

$$D_t^iv_n\to D_t^iv \quad\text{in } \mathcal{E}_t^0(\mathcal{E}^2_{L^2}) \quad\text{as } n\to\infty \quad\text{for } 0\le i\le k$$

Observing equation (7) we can easily conclude that

$$D_t^iD^{3j}v_n\to D_t^iD^{3j}v \quad\text{in } \mathcal{E}_t^0(\mathcal{E}^2_{L^2}) \quad\text{as } n\to\infty \quad\text{for } i+j\le k.$$

Therefore we obtain the following local existence theorem

Theorem 3. *If $f(x)$ and $g(x,t)$ have the same regularity assumptions as that of the main theorem then the Cauchy problem* (5) *has unique solution $v(x,t)$ in $0\le t\le T_k$ which has the same regularity as that of the main theorem.*

Combine this local existence theorem with the global a priori estimate (Theorem 2) we can easily conclude the global existence theorem (Main theorem). This completes the proof of the main theorem.

Remark. Uniqueness of the solutions is easily obtained by the usual method of L^2-energy estimate.

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