# 173. On the Classical Stability Theorem of Poincaré-Lyapunov with a Random Parameter

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1. The objective of this paper is concerned with the generalization of the classical stability theorem of Poincaré-Lyapunov, [1].

The Poincaré-Lyapunov theorem with a random parameter can be written as follows:

$$\dot{x}(t;\omega)=A(\omega)x(t;\omega)+f(t,\ x(t;\omega)),\quad t\geq 0 \tag{1.0}$$

where

( i ) $\omega\in\Omega$, $\Omega$ being the supporting set of the probability measure space $(\Omega, A, \mu)$;

(ii) $x(t;\omega)$ is the unknown $nxl$ random vector;

(iii) $A(\omega)$ is $mxn$ matrix whose elements are measurable functions;

(iv) $f(t, x)$ is for $t\in R_+$ and $x\in R$ an $nxl$ vector valued function.

The above random differential system can be easily reduced into the following stochastic equation

$$x(t;\omega)=e^{A(\omega)t}x_0(\omega)+\int_0^t e^{A(\omega)(t-\tau)}f(\tau,\ x(\tau;\omega))d\tau. \tag{1.1}$$

**Remark.** The term $e^{A(\omega)t}x_0(\omega)$ is referred to as the free stochastic term or free random variable, $e^{A(\omega)(t-\tau)}$ the stochastic kernel and $x(0,\omega)=x_0(\omega)$.

The particular aim of this paper is the existence, uniqueness and asymptotic behavior of a random solution of the stochastic integral equation (1.1). In accomplishing this objective we utilized certain aspects and methods of "admissibility theory" which can be found in [2].

2. We shall consider that the random solution $x(t;\omega)$ and the stochastic free term $e^{A(\omega)t}x_0(\omega)$ are functions of the real argument $t$ with values in the space $L_2(\Omega, A, \mu)$. The function $f(t, x(t;\omega))$, under convenient conditions, will also be a function of $t$ with values in $L_2(\Omega, A, \mu)$. The value of the stochastic kernel, $e^{A(\omega)(t-\tau)}$, $0\leq\tau\leq t$, shall be an essentially bounded function with respect to $\mu$ for every $t$ and $\tau$, such that $0\leq\tau\leq t<\infty$. The values of this term for fixed $t$ and $\tau$, will be in $L_\infty(\Omega, A, \mu)$ so that the product of $e^{A(\omega)t}x_o(\omega)$ and $e^{A(\omega)(t-\tau)}$ will always be in $L_2(\Omega, A, \mu)$.

The norm of the stochastic kernel of the random integral equation

(1.1) will be given by

$$|||e^{A(\omega)(t-\tau)}||| = ||e^{A(\omega)(t-\tau)}||\, L_\infty(\Omega, A, \mu)$$
$$= \mu - \text{ess sup } |e^{A(\omega)(t-\tau)}|,$$

$\omega \in \Omega$. That is, for fixed $t$ and $\tau$,

$$|||e^{A(\omega)(t-\tau)}||| = \inf_{\Omega_0} \{ \sup_{\Omega - \Omega_0} |e^{A(\omega)(t-\tau)}| \},$$

$\mu(\Omega_0) = 0$.

**Definition 2.1.** We shall denote by

$$E_g = E_g(R_+, L_2(\Omega, A, \mu))$$

the Banach space of all continuous functions from $R_+$ into $L_2(\Omega, A, \mu)$, such that

$$\left\{ \int_\Omega |x(t;\omega)|^2 \, d\mu(\omega) \right\}^{1/2} \leq A g(t),$$

where $A$ is a positive number and $g(t)$ is a positive continuous function on $R_+$.

The norm in the space $E_g$ is defined by

$$||x(t;\omega)||\, E_g = \sup_{t \in R_+} \left\{ \frac{1}{g(t)} ||x(t;\omega)|| \right\}$$

where

$$||x(t;\omega)|| = ||x(t;\omega)||_{L_2(\Omega, A, \mu)}$$
$$= \sup_{0 \leq t} \left\{ \int_\Omega |x(t;\omega)|^2 d\mu(\omega) \right\}^{1/2}.$$

**Definition 2.2.** The pair of spaces $(E_1, E_2)$ will be called *admissible* with respect to the operator

$$T : E_g(R_+, L_2(\Omega, A, \mu)) \rightarrow E_g(R_+, L_2(\Omega, A, \mu)),$$

if and only if $TE_1 \subset E_2$.

**Definition 2.3.** $x(t;\omega)$ will be called a *random solution* of the random integral equation (1.1) if for every fixed $t$ belonging to $R_+$, $x(t;\omega) \in L_2(\Omega, A, \mu)$ and satisfies equation (1.1) $\mu - a.e.$

**Definition 2.4.** The random solution $x(t;\omega)$ is said to be *stochastically asymptotically exponentially stable* if there exists a $\rho > 0$, such that,

$$\left\{ \int_\Omega |x(t;\omega)|^2 d\mu(\omega) \right\}^{1/2} \leq \rho e^{-\beta t},$$

where $\beta > 0$.

Finally, for $E_1$ and $E_2$ a pair of Banach spaces and $T$ a linear operator, we state the following lemma which will be used in the main theorem of this paper.

**Lemma 2.1.** *Let $T$ be a continuous operator from $E_g(R_+, L_2(\Omega, A, \mu))$ into itself. If $E_1$ and $E_2$ are Banach spaces stronger than $E_g$ and the pair $(E_1, E_2)$ is admissible with respect to $T$, then $T$ is a continuous operator from $E_1$ to $E_2$.*

The lemma follows easily from the closed graph theorem.

**Remark.** Since $T$ is a continuous operator it is also bounded.

Then it follows that we can find a constant $K>0$, such that

$$\|(Tx)(t;\omega)\|_{E_2}\le K\|x(t;\omega)\|_{E_1}.$$

**3.** With respect to the aim of this paper we state and prove the following theorem.

**Theorem 3.1.** *Let us assume that the random integral equation* (1.1) *satisfies the following conditions*:

(i) *The matrix $A(\omega)$ is stochastically stable, that is, there exists $\alpha>0$, such that*

$$\mu\{\omega;\operatorname{Re}\psi_k(\omega)<-\alpha,\ k=1,2,\cdots,n\}=1,$$

*where $\psi_k(\omega)$, $k=1,2,\cdots,n$, are the characteristic roots of the matrix*;

(ii) *$x(t;\omega)\to f(t,x(t;\omega))$ is an operator on*

$$S=\{x(t;\omega);\ x(t;\omega)\in E_g,\ \|x(t;\omega)\|_{E_g\le\rho}\},$$

*with values in $E_g$ satisfying*

$$\|f(t,x(t;\omega))-f(t,y(t;\omega))\|_{E_g}\le\lambda\|x(t;\omega)\|_{E_g}$$

*for $x(t;\omega)$, $y(t;\omega)\in S$, $\lambda$ being a constant and $f(t,0)=0$.*

Then, there *exists* a *unique* random solution of the random integral equation (1.1), such that

$$\lim_{t\to\infty}\left\{\int_\Omega|x(t;\omega)|^2d\mu(\omega)\right\}^{1/2}=0,$$

provided that

$$\lambda<K^{-1},\ \|e^{A(\omega)t}x_0(\omega)\|_{E_g}\le\rho(1-\lambda K)$$

where $K$ is the norm of the operator $T$.

**Proof.** First we will show that the pair of Banach spaces $(E_g, E_g)$ with $g(t)=e^{-\beta t}$, where $0<\beta<\alpha$, is admissible under the above conditions. Recall that the norm in $E_g$ space is defined by

$$\|x(t;\omega)\|_{E_g}=\sup_{t\in R_+}\frac{1}{g(t)}\left\{\int_\Omega|x(t;\omega)|^2d\mu(\omega)\right\}^{1/2},$$

and for $x(t;\omega)\in E_g$ let us define the following integral operator

$$(Tx)(t;\omega)=\int_0^t e^{A(\omega)(t-\tau)}x(\tau;\omega)d\tau.$$

It follows that

$$\|(Tx)(t;\omega)\|\le\int_0^t e^{A(\omega)(t-\tau)}\|x(\tau;\omega)\|d\tau. \tag{3.1}$$

It has been shown by T. Morozan [3] that there exists a subset $\Omega_0$ of $\Omega$, such that, $\mu(\Omega_0)=1$ and

$$|||e^{A(\omega)(t-\tau)}|||\le Me^{-\alpha(t-\tau)}, \tag{3.2}$$

for $\omega\in\Omega_0$, $K>0$ and $\alpha$ as defined above. Applying inequality (3.2) to inequality (3.1) we have

$$\|(Tx)(t;\omega)\|\le M\int_0^t e^{-\alpha(t-\tau)}\frac{\|x(\tau;\omega)\|}{g(\tau)}g(\tau)d\tau. \tag{3.3}$$

Since we have chosen $g(t)=e^{-\beta t}$, $0<\beta<\alpha$, inequality (3.3) can be written as

$$\begin{aligned}\|(Tx)(t;\omega)\| &\leq M\int_0^t e^{-\alpha t}e^{\tau(\alpha-\beta)}\frac{\|x(t;\omega)\|}{e^{-\beta\tau}}d\tau\\ &\leq M\|x(t;\omega)\|_{E_g}e^{-\alpha t}\int_0^t e^{\tau(\alpha-\beta)}d\tau\\ &\leq M\|x(t;\omega)\|_{E_g}(\alpha-\beta)^{-1}[e^{-\beta t}-e^{-\alpha t}],\quad t\leq 0.\end{aligned}\tag{3.4}$$

Since $0<\beta<\alpha$, inequality (3.4) can be majorized as follows:

$$\|(Tx)(t;\omega)\|\leq M\|x(t;\omega)\|_{E_g}(\alpha-\beta)^{-1}e^{-\beta t}$$

from which it follows that

$$\begin{aligned}\|(Tx)(t;\omega)\|_{E_g} &\leq M(\alpha-\beta)^{-1}\|x(t;\omega)\|_{E_g}\\ &\leq K\|x(t;\omega)\|_{E_g}.\end{aligned}$$

Hence, $x(t;\omega)\in E_g$ implies that $TE_g\subset E_g$, which implies that the pair of Banach spaces $(E_g, E_g)$ is admissible.

Now, let us define an operator $U$ from $S$ into $E_g$ as follows:

$$(Ux)(t;\omega)=e^{A(\omega)t}x_0(\omega)+\int_0^t e^{A(\omega)(t-\tau)}f(\tau, x(\tau;\omega))d\tau.\tag{3.5}$$

We must show that $U$ is a contracting operator and $US\subset S$. Consider an element $y(t;\omega)\in S$. We can write

$$(Uy)(t;\omega)=e^{A(\omega)t}x_0(\omega)+\int_0^t e^{A(\omega)(t-\tau)}f(\tau, y(\tau;\omega))d\tau.\tag{3.6}$$

Subtracting equation (3.6) from equation (3.5) we have

$$(Ux)(t;\omega)-(Uy)(t;\omega)=\int_0^t e^{A(\omega)(t-\tau)}[f(\tau, x(\tau;\omega))-f(\tau, y(\tau;\omega))]d\tau.$$

Since $US\subset E_g$ is a Banach space, then

$$(Ux)(t;\omega)-(Uy)(t;\omega)\in E_g.$$

By assumption (ii), $[f(t, x(t;\omega))-f(t, y(t;\omega))]\in E_g$. From Lemma 2.1 we have seen that $T$ is a continuous operator from the Banach space $E_g$ into $E_g$, which implies that we can find a constant $K>0$, such that,

$$\|(Tx)(t;\omega)\|_{E_g}\leq K\|x(t;\omega)\|_{E_g}.$$

That is,

$$\|(Ux)(t;\omega)-(Uy)(t;\omega)\|_{E_g}\leq K\|f(t, x(t;\omega))-f(t, y(t;\omega))\|_{E_g}.$$

Now, applying Lipschitz's condition given in (ii) we have

$$\|(Ux)(t;\omega)-(Uy)(t;\omega)\|_{E_g}\leq \lambda K\|x(t;\omega)-y(t;\omega)\|_{E_g}.$$

Applying the condition that $\lambda K<1$, it implies that the operator $U$ is a contracting operator. It now remains to be shown that $US\subset S$. For every $x(t;\omega)\in S$, we have

$$(Ux)(t;\omega)=e^{A(\omega)t}x_0(\omega)+\int_0^t e^{A(\omega)(t-\tau)}f(\tau, x(\tau;\omega))d\tau.\tag{3.7}$$

It follows that

$$\|(Ux)(t;\omega)\|_{E_g}\leq\|e^{A(\omega)t}x_0(\omega)\|+\int_0^t |||e^{A(\omega)(t-\tau)}|||\ \|f(\tau, x(\tau;\omega))\|d\tau\tag{3.8}$$

but, $|||e^{A(\omega)(t-\tau)}|||\leq Me^{-\alpha(t-\tau)}$, which implies that inequality (3.8) can be written as

$$\|(Ux)(t;\omega)\|_{E_g} \leq \|e^{A(\omega)t}x_0(\omega)\| + M\int_0^t e^{-\alpha(t-\tau)}\frac{1}{g(\tau)}\|f(\tau, x(\tau;\omega))\|\,g(\tau)d\tau. \tag{3.9}$$

Since $g(t)=e^{-\beta t}$, (3.9) becomes

$$\begin{aligned}\|(Ux)(t;\omega)\|_{E_g} &\leq \|e^{A(\omega)t}x_0(\omega)\| + M\|f(t, x(t,\omega))\|_{E_g}\int_0^t e^{-\alpha(t-\tau)}g(\tau)d\tau \\ &\leq \|e^{A(\omega)t}x_0(\omega)\| + M\|f(t, x(t;\omega))\|_{E_g}e^{-\alpha t}\int_0^t e^{(\alpha-\beta)}\,d\tau \\ &\leq \|e^{A(\omega)t}x_0(\omega)\| + M(\alpha-\beta)^{-1}\|f(t, x(t;\omega))\|_{E_g}.\end{aligned} \tag{3.10}$$

By adding and subtracting $f(t, 0)$ and applying Lipschitz's condition inequality (3.10) becomes

$$\|(Ux)(t;\omega)\|_{E_g} \leq \|e^{A(\omega)t}x_0(\omega)\| + K\lambda\|x(t;\omega)\|_{E_g}. \tag{3.11}$$

Since $x(t;\omega)\in S$ and $\|x(t;\omega)\|_{E_g}\leq\rho$ together with the condition that $\|e^{A(\omega)t}x_0(\omega)\|\leq\rho(1-\lambda K)$, equation (3.11) reduces to

$$\|(Ux)(t;\omega)\|\leq\rho(1-\lambda K)+K\lambda\rho=\rho,$$

which implies that $(Ux)(t;\omega)\in S$ for all $x(t;\omega)\in S$, or $US\subset S$. Therefore, since $U$ is a contracting operator and $US\subset S$ (Inclusion property), applying Banach's Fixed Point Theorem, there *exists* a unique random solution of the random differential system (1.0) which is *exponentially stochastically stable*.

## References

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[3] Morozan, T.: Stability of linear systems with random parameters. Journal of Differential Equations, **3**, 170–178 (1967).