171. *On the Nörlund Summability of Fourier Series*

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1. Introduction and Theorems.

1.1. Definitions. Let $\sum a_n$ be a given series and s_n be its nth partial sum. Let (p_n) be a sequence of real numbers such that $p_0=0$, $P_n=p_0+p_1+\cdots+p_n\neq 0$ for all n and $|P_n|\to\infty$ as $n\to\infty$. If the sequence

$$t_n=\frac{1}{P_n}\sum_{k=0}^{n} p_{n-k}s_k \qquad (n=1, 2, \cdots) \tag{1}$$

tends to a limit s as $n\to\infty$, then the series $\sum a_n$ is said to be (N, p_n) summable to s. This method of summation is regular if and only if

$$\sum_{k=0}^{n}|p_k|\leqq A|P_n| \qquad \text{for all } n\geqq 1 \text{ and } p_n/P_n\to 0 \text{ as } n\to\infty. \tag{2}$$

Let f be an integrable function with period 2π and its Fourier series be

$$f(t)\sim\frac{1}{2}a_0+\sum_{n=1}^{\infty}(a_n\cos nt+b_n\sin nt)=\sum_{n=0}^{\infty}A_n(x). \tag{3}$$

We write $\varphi(t)=\varphi_x(t)=f(x+t)+f(x-t)-2f(x)$.

1.2. E. Hille and J. D. Tamarkin [1] have applied the (N, p_n) summation to Fourier series. Extending one of their theorems, O. P. Vershney [2] has proved the following

Theorem I. *Suppose that the sequence (p_n) of real numbers satisfies the conditions*:

$$n|p_n|\leqq A|P_n|\log(n+1) \qquad \textit{for } n\geqq 1, \tag{4}$$

$$\sum_{k=1}^{n}\frac{k|\Delta p_k|}{\log(k+1)}\leqq A|P_n| \qquad \textit{for all } n\geqq 1 \tag{5}$$

and

$$\sum_{k=1}^{n}\frac{|P_k|}{k\log(k+1)}\leqq A|P_n| \qquad \textit{for all } n\geqq 1. \tag{6}$$

If

$$\Phi(t)=\int_0^t|\varphi(u)|\,du=o\left(t/\log\frac{1}{t}\right) \qquad \textit{as } t\to 0, \tag{7}$$

then the Fourier series of f is (N, p_n) summable to $f(x)$.

On the other hand O. P. Vershney [3] proved the

Theorem II. *Let (p_n) be a positive non-increasing sequence. Then the Fourier series of f satisfying the condition*

$$\varphi(t)=o\left(1/\log\frac{1}{t}\right) \qquad \textit{as } t\to 0,$$

is (N, p_n) *summable at the point* x, *if and only if*

$$\sum_{k=1}^{n} \frac{P_k}{k \log (k+1)} \leqq AP_n \qquad \text{for all } n \geqq 1.$$

We are going to prove a theorem containing both of above theorems:

Theorem 1. *Suppose that* $L(u)$ *is a positive non-decreasing function on the interval* $(0, \infty)$, $k/L(k) \uparrow \infty$ *as* $k \to \infty$ *and* $\int_1^k \frac{du}{L(u)} \leqq \frac{Ak}{L(k)}$ *for all* $k \geqq 1$ *and that the sequence* (p_n) *of real numbers satisfies the condition*

$$(8) \qquad \sum_{k=1}^{n} \frac{k|\Delta p_k|}{L(k)} \leqq A|P_n| \qquad \text{for all } n \geqq 1.$$

(i) *If*

$$(9) \qquad \sum_{k=1}^{n} \frac{|P_k|}{kL(k)} \leqq A|P_n| \qquad \text{for all } n \geqq 1,$$

then the Fourier series of f *satisfying the condition*

$$(10) \qquad \Phi(t) = \int_0^t |\varphi(u)|\, du = o(t/L(1/t)) \qquad \text{as } t \to 0,$$

is (N, p_n) *summable to* $f(x)$ *at the point* x.

(ii) *If* $P_n > 0$ *for all* n, *then the condition* (9) *is a necessary and sufficient condition for* (N, p_n) *summability of* f *satisfying the condition* (10).

If $L(u) = \log(u+1)$ in Theorem 1, this is a generalization of Theorems I and II.

The case $L(u) = (\log(u+1))^a$ $(0 < a < 1)$ was treated by B.N. Sahney [4] for f satisfying stronger condition.

The condition (8) implies

$$(11) \qquad n|p_n| \leqq A|P_n|L(n) \quad \text{and} \quad n|p_{n+1}| \leqq A|P_n|L(n) \qquad \text{for all } n \geqq 1$$

since

$$A|P_n| \geqq \sum_{k=1}^{n} \frac{k|\Delta p_k|}{L(k)} \geqq \frac{1}{L(n)} \left| \sum_{k=1}^{n-1} k\Delta p_k \right| = \frac{|P_n - np_n|}{L(n)}.$$

1.3. We define a function $p(u)$ on the interval $(0, \infty)$ such that $p(n) = p_n$ for every integer $n \geqq 0$ and $p(u)$ is linear at every non-integral point u and is continuous in the whole interval. We put $P(u) = \int_0^u p(v)dv$ for $u > 0$, then $P(n) = \sum_{k=1}^{n-1} p_k + \frac{1}{2} p_n$.

We have proved the following theorem [5], as a generalization of the T. Singh theorem [6]:

Theorem III. *If* (p_n) *is a positive sequence satisfying the condition*

$$(12) \qquad \sum_{k=1}^{n} k|\Delta p_k| \leqq AP_n \qquad \text{for all } n \geqq 1$$

and if

(13) $$\Phi(t)=o(|p(1/t)|/|P(1/t)|) \qquad as\ t\to 0,$$

then the Fourier series of f is (N, p_n) summable to $f(x)$ at the point x.

We can generalize this theorem as follows:

Theorem 2. *Suppose that (p_n) is a sequence of real numbers satisfying the condition*

(14) $$\sum_{k=1}^{n} k|\Delta p_k| \leqq A|P_n| \qquad for\ all\ n\geqq 1,$$

then the Fourier series of f satisfying the condition (13) *is (N, p_n) summable to $f(x)$ at the point x.*

The condition (14) implies that $n|p_n|\leqq A|P_n|$ and $n|p_{n+1}|\leqq A|P_n|$ for all $n\geqq 1$.

2. Proof of Theorem 1. By (1), the nth Nörlund mean of the Fourier series is

$$t_n=\frac{1}{P_n}\sum_{k=0}^{n} p_{n-k}\int_0^\pi \varphi(t)\frac{\sin(k+1/2)t}{2\sin t/2}dt$$

$$=-\frac{1}{P_n}\sum_{k=0}^{n} p_{n-k}\int_0^\pi \frac{\varphi(t)\cos(n+1/2)t}{2\sin t/2}\sin(n-k)t\,dt$$

$$+\frac{1}{P_n}\sum_{k=0}^{n} p_{n-k}\int_0^\pi \frac{\varphi(t)\sin(n+1/2)t}{2\sin t/2}\cos(n-k)t\,dt \;=-u_n+\bar{u}_n.$$

We shall estimate u_n. We write $h(t)=\varphi(t)\cos(n+1/2)t$, then, by (10) and (11)

$$u_n=\frac{1}{P_n}\int_0^\pi \frac{h(t)}{(2\sin t/2)^2}\Big\{\sum_{k=1}^{n-1}\Delta p_k(1-\cos(k+1/2)t)$$

$$-p_1(1-\cos t/2)+p_n(1-\cos(n+1/2)t)\Big\}dt$$

$$=\frac{1}{P_n}\int_0^\pi \frac{h(t)}{(2\sin t/2)^2}\Big\{\sum_{k=1}^{n-1}\Delta p_k(1-\cos(k+1/2)t)\Big\}dt+o(1)$$

$$=\frac{1}{P_n}\sum_{k=1}^{n-1}\Delta p_k\left(\int_0^{1/k}+\int_{1/k}^{\pi}\right)\frac{h(t)}{(2\sin t/2)^2}(1-\cos(k+1/2)t)dt+o(1)$$

$$=v_n+w_n+o(1) \qquad \text{as } n\to\infty,$$

where

$$|v_n|\leqq\frac{1}{|P_n|}\sum_{k=1}^{n-1}k^2|\Delta p_k|\int_0^{1/k}|\varphi(t)|dt=o(1) \qquad \text{as } n\to\infty$$

by the conditions (8) and (10), and

$$|w_n|\leqq\frac{1}{|P_n|}\sum_{k=1}^{n-1}|\Delta p_k|\int_{1/k}^{\pi}\frac{|\varphi(t)|}{t^2}dt=o(1) \qquad \text{as } n\to\infty,$$

since

$$\int_{1/k}^{\pi}\frac{|\varphi(t)|}{t^2}dt=\left[\frac{\Phi(t)}{t^2}\right]_{1/k}^{\pi}+2\int_{1/k}^{\pi}\frac{\Phi(t)}{t^2}dt=o\left(\frac{k}{L(k)}\right) \qquad \text{as } k\to\infty$$

by (10). Thus we have proved that $u_n=o(1)$ as $n\to\infty$.

Putting $\bar{h}(t)=\varphi(t)\sin(n+1/2)t$, we have

$$\bar{u}_n=\frac{1}{P_n}\int_0^\pi \frac{\bar{h}(t)}{(2\sin t/2)^2}\left\{\sum_{k=1}^{n-1} \Delta p_k \sin(k+1/2)t - p_1\sin t/2 + p_n \sin(n+1/2)t\right\}dt$$

$$=\frac{1}{P_n}\int_0^\pi \frac{\bar{h}(t)}{(2\sin t/2)^2}\left\{\sum_{k=1}^{n-1} \Delta p_k \sin(k+1/2)t\right\}dt+o(1)$$

$$=\frac{1}{P_n}\sum_{k=1}^{n-1}\Delta p_k\left(\int_0^{1/k}+\int_{1/k}^\pi\right)\frac{\bar{h}(t)}{(2\sin t/2)^2}\sin(k+1/2)t\,dt+o(1)$$

$$=\bar{v}_n+\bar{w}_n+o(1),$$

where $\bar{w}_n=o(1)$ by similar estimation to w_n. Now, $\bar{v}_n$ has a different feature from v_n. We have

$$\bar{v}_n=\frac{1}{P_n}\sum_{k=1}^{n-1}\Delta p_k\left(\int_0^{1/n}+\int_{1/n}^{1/k}\right)\frac{\bar{h}(t)}{t^2}\sin kt\,dt+o(1)=\bar{x}_n+\bar{y}_n+o(1)$$

where

$$|\bar{x}_n|\leqq\frac{n}{|P_n|}\sum_{k=1}^{n-1}k|\Delta p_k|\int_0^{1/n}|\varphi(t)|\,dt$$

$$=o\left(\frac{1}{|P_n|L(n)}\sum_{k=1}^{n-1}k|\Delta p_k|\right)=o(1)\qquad\text{as } n\to\infty$$

by (8) and (10). Again using (8) and (10), we get

$$P_n\bar{y}_n=\sum_{k=1}^{n-1}\Delta p_k\int_{1/n}^{1/k}\bar{h}(t)t^{-2}(kt+O(k^3t^3))dt$$

$$=\sum_{k=1}^{n-1}k\Delta p_k\sum_{j=k}^{n-1}\int_{1/(j+1)}^{1/j}\bar{h}(t)t^{-1}\,dt+o(|P_n|)$$

$$=\sum_{j=1}^{n-1}\int_{1/(j+1)}^{1/j}\bar{h}(t)t^{-1}\,dt\left(\sum_{k=1}^{j}k\Delta p_k\right)+o(|P_n|)$$

$$=\sum_{j=1}^{n-1}\int_{1/(j+1)}^{1/j}\frac{\bar{h}(t)}{t}\left(P\left(\frac{1}{t}\right)-\frac{1}{t}p\left(\frac{1}{t}\right)\right)dt+o(|P_n|),$$

since

$$\sum_{k=1}^{j}k\Delta p_k=P_{j+1}-(j+1)p_{j+1}$$

$$=P\left(\frac{1}{t}\right)-\frac{1}{t}p\left(\frac{1}{t}\right)+O(|p_j|+|p_{j+1}|+j|\Delta p_j|)$$

and

$$\sum_{j=1}^{n-1}j|p_j|\int_{1/(j+1)}^{1/j}|\varphi(t)|\,dt=o\left(\sum_{j=1}^{n-1}\frac{|p_j|}{L(j)}\right)=o(|P_n|),$$

$$\sum_{j=1}^{n-1}j^2|\Delta p_j|\int_{1/(j+1)}^{1/j}|\varphi(t)|\,dt=o\left(\sum_{j=1}^{n-1}\frac{j|\Delta p_j|}{L(j)}\right)=o(|P_n|)$$

by (8), (10) and (11). Therefore, putting $\bar{H}(t)=\int_0^t\bar{h}(u)\,du$, we get

$$P_n\bar{y}_n=\int_{1/n}^1\frac{\bar{h}(t)}{t}\left(P\left(\frac{1}{t}\right)-\frac{1}{t}p\left(\frac{1}{t}\right)\right)dt+o(|P_n|)$$

$$=\left[\frac{\bar{H}(t)}{t}\left(P\left(\frac{1}{t}\right)-\frac{1}{t}p\left(\frac{1}{t}\right)\right)\right]_{t=1/n}^{1}+\int_{1/n}^{1}\frac{\bar{H}(t)}{t^2}P\left(\frac{1}{t}\right)dt$$

$$-\int_{1/n}^{1}\frac{\bar{H}(t)}{t^3}p\left(\frac{1}{t}\right)dt-\int_{1/n}^{1}\frac{\bar{H}(t)}{t^4}p'\left(\frac{1}{t}\right)dt+o(|P_n|)$$

$$=\int_{1/n}^{1}\frac{\bar{H}(t)}{t^2}P\left(\frac{1}{t}\right)dt+o(|P_n|) \qquad \text{as } n\to\infty.$$

Collecting above estimations, we get

$$(15)\qquad t_n=\frac{1}{\pi P_n}\int_{1/n}^{1}\frac{\bar{H}(t)}{t^2}P\left(\frac{1}{t}\right)dt+o(1) \qquad \text{as } n\to\infty.$$

By (9) and (10), we get $t_n=o(1)$ as $n\to\infty$. Hence the condition (9) is sufficient.

We shall now prove the necessity of the condition (9), supposing that $P_n\geqq 0$. We suppose that $t_n\to 0$ as $n\to\infty$. If the condition (9) does not hold, then there are an increasing sequence (n_k) of integers and an increasing sequence (C_k) tending to infinity such that

$$(16)\qquad \frac{1}{P_{n_k}}\int_{1/n_k}^{1}\frac{P(1/t)}{tL(1/t)}dt>C_k \qquad (k=1, 2, \cdots).$$

We can find a function φ_0 such that

$$(17)\qquad \int_0^t|\varphi_0(u)|\,du=\varepsilon(t).\quad t/L(1/t) \text{ for } t>0,$$

where $\varepsilon(t)\downarrow 0$ as $t\downarrow 0$ and $\varepsilon(1/n_k)=1/\sqrt{C_k}$ for all k. We define (m_k), a subsequence of (n_k), by the inductive method as follows: Let $m_1=n_1$, and if m_k is defined, m_{k+1} is taken such that

$$(18)\qquad \int_{1/m_{k+1}}^{1/m_k}|\varphi_0(u)|\,du>2\int_0^{1/m_{k+1}}|\varphi_0(u)|\,du$$

and

$$\frac{1}{P_{m_k}}\int_{1/m_k}^{1/m_{k-1}}\frac{P(1/t)}{tL(1/t)}dt\geqq\frac{1}{2}C_{m_k},\quad \frac{1}{P_{m_k}}\int_{1/m_{k-1}}^{1}\frac{P(1/t)}{tL(1/t)}dt<\frac{1}{16}\sqrt{C_k}.$$

We shall define a function φ such that

$\varphi(u)\sin(m_{2k}+1/2)u=|\varphi_0(u)|$ for $(1/m_{2k+1}, 1/m_{2k-1})$ $\qquad (k=1, 2, \cdots)$.

Then, by (17) and (18),

$$\frac{1}{P_{m_{2k}}}\int_{1/m_{2k}}^{1}\frac{P(1/t)}{t^2}dt\int_0^t\varphi(u)\sin(m_{2k}+1/2)u\,du$$

$$\geqq\frac{1}{P_{m_{2k}}}\int_{1/m_{2k}}^{1/m_{2k-1}}\frac{P(1/t)}{t^2}dt\left(\int_{1/m_{2k+1}}^{t}\varphi(u)\sin(m_{2k}+1/2)u\,du\right.$$

$$\left.-\int_0^{1/m_{2k+1}}|\varphi(u)|\,du\right)-\frac{1}{P_{m_{2k}}}\int_{1/m_{2k-1}}^{1}\frac{P(1/t)}{t^2}\int_0^t|\varphi(u)|\,du$$

$$\geqq\frac{1}{P_{m_{2k}}}\left(\frac{1}{2}\int_{1/m_{2k}}^{1/m_{2k-1}}\frac{P(1/t)}{t^2}dt\int_{1/m_{2k+1}}^{t}|\varphi_0(u)|\,du-\int_{1/m_{2k-1}}^{1}\frac{P(1/t)}{t^2}dt\right.$$

$$\left.\times\int_0^t|\varphi_0(u)|\,du\right)$$

$$\geqq \frac{1}{P_{m_{2k}}}\left(\frac{1}{4}\int_{1/m_{2k}}^{1/m_{2k-1}} \frac{P(1/t)}{t^2}dt\int_0^t |\varphi_0(u)|\,du - \int_{1/m_{2k-1}}^{1} \frac{P(1/t)}{t^2}dt\int_0^t |\varphi_0(u)|\,du\right)$$
$$\geqq \frac{1}{P_{m_{2k}}}\left(\frac{1}{4}\int_{1/m_{2k}}^{1/m_{2k-1}} \frac{P(1/t)\varepsilon(t)}{tL(1/t)}dt - \int_{1/m_{2k-1}}^{1} \frac{\varepsilon(t)P(1/t)}{tL(1/t)}dt\right)$$
$$\geqq \frac{1}{P_{m_{2k}}}\left(\frac{\varepsilon(1/m_{2k})}{4}\int_{1/m_{2k}}^{1/m_{2k-1}} \frac{P(1/t)}{tL(1/t)}dt - \int_{1/m_{2k-1}}^{1} \frac{P(1/t)}{tL(1/t)}dt\right)$$
$$\geqq \frac{1}{8}C_{m_{2k}}/\sqrt{C_{m2k}} - \frac{1}{16}\sqrt{C_{m_{2k}}} = \frac{1}{16}\sqrt{C_{m_{2k}}} \to \infty \qquad \text{as } k\to\infty,$$

which is a contradiction.

3. Proof of Theorem 2. Proof runs similarly to that of Theorem 1, so that we shall only remark the following facts. By the conditions (13) and (14), we get

$$\int_{1/k}^{\pi} |\varphi(t)|\,t^{-2}dt = [\Phi(t)t^{-2}]_{1/k}^{1} + 2\int_{1/k}^{1} \Phi(t)t^{-3}\,dt$$
$$\leqq A + o(k) + o\left(\int_{1/k}^{1} \frac{|p(1/t)|}{t^3P(1/t)}dt\right) = o(k),$$
$$\int_{1/n}^{\pi} \frac{\Phi(t)}{t^4}\,p'\left(\frac{1}{t}\right)dt = o\left(\int_{1/n}^{1} \frac{|p(1/t)p'(1/t)|}{t^4P(1/t)}dt\right)$$
$$= o\left(\sum_{k=1}^{n} k^2|p_k\cdot \Delta p_k|/|P_k|\right) = o\left(\sum_{k=1}^{n} k|\Delta p_k|\right) = o(|P_n|),$$

and then we can use them for the estimation of w_n and $P_n\bar{y}_n$, respectively. The integral of (15) is $o\left(\sum_{k=1}^{n}|p_k|\right) = o\left(\sum_{k=1}^{n} k|\Delta p_k| + n|p_{n+1}|\right) = o(|P_n|)$. Thus Theorem 2 is proved.

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