197. Necessary and Sufficient Conditions for the Normality of the Product of Two Spaces

By Masahiko ATSUJI
Department of Mathematics, Josai University, Saitama

In this paper we shall present a solution (Theorem) of the problem to find necessary and sufficient conditions for the normality of a product space $X \times Y$.

This fundamental problem has been researched by many mathematicians probably since about the time (1925) when the importance of normal spaces was found by P. Urysohn [1]. Though the problem has been unsettled for a fairly long time, the proof of our Theorem is simple and elementary. Difficulty may have been in the formulation of the theorem, which is natural but apparently pretty different from ones conjectured from known partial solutions.

In this paper a space is, unless otherwise specified, topological. Let $A$ be a subset of the product space $X \times Y$ of spaces $X$ and $Y$, then we write for $x \in X$

$$A[x] = \{y \in Y; (x, y) \in A\}.$$

Definition 1. Let $\mathfrak{F}$ be a family of neighborhoods of $a \in X$, and let $\{A_x; x \in Z \subset X\}$ any family of subsets $A_x$ of $Y$, then we write

$$\limsup_{\mathfrak{F}} A_x = \bigcap_{U \in \mathfrak{F}} \bigcup_{x \in U} A_x,$$

$$c-lim\sup_{\mathfrak{F}} A_x = Y - \limsup_{\mathfrak{F}} (Y - A_x),$$

where the bar means the closure in $Y$ and $x \in U$ does $x \in U \cap Z$.

Hereafter, let us denote by $\mathcal{U}_a$ for $a \in X$ the neighborhood system of $a$ in $X$, and we write “lim sup” instead of “lim sup”. We can easily obtain

Proposition 1. Let $\mathfrak{U}$ be a neighborhood base of $a$ in $X$, then

$$\limsup_{\mathfrak{U}} A_x = \limsup_a A_x,$$

$$c-lim\sup_{\mathfrak{U}} A_x = c-lim\sup_a A_x.$$

Proposition 2. Let $\{A_x; x \in Z \subset X\}$ be any family of sets $A_x \subset Y$, and put

$$(x, A_x) = \{(x, y); y \in A_x\},$$

$$A = \bigcup_{x \in Z} (x, A_x),$$

then
(i) $\bar{A}[a] = \limsup_a A[x] = \limsup_a A_x$, 

(ii) $A^a[a] = c\limsup_a A[x] = c\limsup_a A_x$

for any $a \in X$, where the bar and 0 mean the closure and the interior in $X \times Y$ respectively.

Proof of (i) is obtained by the following equivalent statements.

1. $p \in \bar{A}[a]$.
2. $(a, p) \in \bar{A}$.
3. $(U \times V) \cap A \neq \emptyset$ for any $U \in \mathcal{R}_a$ and $V \in \mathcal{R}_p$.
4. $V \cap A[x] \neq \emptyset$ for any $U \in \mathcal{R}_a$, $V \in \mathcal{R}_p$ and some $x \in U$.
5. $V \cap \{ \bigcup_{x \in U} A[x] \} \neq \emptyset$ for any $U \in \mathcal{R}_a$ and $V \in \mathcal{R}_p$.

The proof of (ii) is obtained by this:

$c\limsup_a A[x] = \bigcap_{x \in U} A[x]$.

The proof of (ii) is obtained by this:

$\bigcup_{x \in U} \bar{A}[a]$ for any $U \in \mathcal{R}_a$.

Corollary 1. A set $A$ in $X \times Y$ is closed (open) if and only if

$\limsup_a A[x] = A[a]$ 

$(c\limsup_a A[x] = A[a])$

for any $a \in X$.

Corollary 2. Let $\{A_z; x \in Z \subset X\}$ be any family of sets $A_z \subset Y$, then

$\limsup_a A[x] = \limsup (\limsup A_x)$,

$c\limsup_a A_x = c\limsup (c\limsup A_x)$.

Proof. Put

$A = \bigcup_{x \in X} (x, A_x)$,

$B = \bigcup_{x \in X} (x, \limsup A_x)$,

then

$\bar{A}[a] = \limsup_a A_x = B[a]$

for any $a \in X$, so we have

$\bar{A} = B$,

$\limsup (\limsup A_x) = B[a] = B[a] = \limsup A_x$.

Definition 2. The following property is denoted by $P(X, Y)$. Let $\{A_z \subset Y; x \in X\}$ and $\{B_z \subset Y; x \in X\}$ be any families with

$(\ast)$ $\limsup_a A_x \cap \limsup_a B_x = \emptyset$
for any $a \in X$, then there are families $\{G_x \subset Y ; x \in X\}$ and $\{H_x \subset Y ; x \in X\}$ satisfying

(i) $G_x \cap H_x = \emptyset$ for any $x \in X$,
(ii) $\operatorname{c-lim sup} G_x \supset A_a$

and

$\operatorname{c-lim sup} H_x \supset B_a$

for any $a \in X$. $\{G_x ; x \in X\}$ and $\{H_x ; x \in X\}$ are called the separating families (or separators) of $\{A_x ; x \in X\}$ and $\{B_x ; x \in X\}$.

Remark. We consider $A$ and $B$ are defined for every point $x$ of $X$, and they may be empty for some $x$. Considering Corollary 2 above, in Definition 2 we can assume that $A$ and $B$ are closed and $G_x$ and $H_x$ are open. It is convenient to have another formulation:

$\operatorname{c-lim sup} A = \bigcup_{\forall a \in \mathcal{A}} \left( \bigcap_{x \in X} A_x \right)^a$. 

Proposition 3. The property $P(X, Y)$ is equivalent with the following property $P_1(X, Y)$. Let $\{A_x \subset Y ; x \in X\}$ and $\{B_x \subset Y ; x \in X\}$ be any families with

(*) $\operatorname{lim sup} A_x \cap \operatorname{lim sup} B_x = 0$

for any $a \in X$, then there is a family $\{G_x \subset Y ; x \in X\}$ satisfying

(i) $\operatorname{c-lim sup} G_x \supset A_a$

and

(ii) $\operatorname{lim sup} G_x \cap B_a = 0$

for any $a \in X$.

Proof. Suppose that $P(X, Y)$ is satisfied, then there are $\{G_x ; x \in X\}$ and $\{H_x ; x \in X\}$ satisfying (i) and (ii) in Definition 2, and we have

$B_a \subset \operatorname{c-lim sup} H_x \subset \operatorname{c-lim sup} \mathcal{C} G_x = \mathcal{C}(\operatorname{lim sup} G_x)$,

namely,

$\operatorname{lim sup} G_x \cap B_a = 0$.

Assuming conversely $P_1(X, Y)$, we put $H_x = \mathcal{C} G_x$, then

$B_a \subset \mathcal{C}(\operatorname{lim sup} G_x) = \operatorname{c-lim sup} H_x$.

It is said that a space satisfies the separation axiom $T_i$ if any two disjoint closed subsets of the space are separated by two disjoint open subsets.

Proposition 4. If $P(X, Y)$ is satisfied, then $Y$ satisfies the separation axiom $T_i$.

Proof. Let $A$ and $B$ be any two disjoint closed subsets of $Y$. Take a point $c \in X$, and put $A_c = A$ and $B_c = B$, and $A_c = B_c = 0$ for $x \neq c$, then

$\operatorname{lim sup} A_x \cap \operatorname{lim sup} B_x = 0$
for any $a \in X$, so there are separating families $\{G_x \subset Y; x \in X\}$ and $\{H_x \subset Y; x \in X\}$ of $\{A_x\}$ and $\{B_x\}$. $T_i$ follows from

$$G^a \supset c\limsup_{x} G_x \supset A_x,$$

$$H^a \supset c\limsup_{x} H_x \supset B_x.$$

Now we can prove our main theorem.

**Theorem.** One of the properties $P(X, Y)$ and $P(Y, X)$ is necessary and sufficient in order that the product space $X \times Y$ satisfies the separation axiom $T_i$.

**Proof.** Necessity. Suppose that $X \times Y$ satisfies $T_i$ and that $\{A_x \subset Y; x \in X\}$ and $\{B_x \subset Y; x \in X\}$ fulfil

$$\limsup_{a} A_x \cap \limsup_{a} B_x = 0$$

for any $a \in X$. Put

$$A = \bigcup_{x \in X} (x, A_x)$$

and

$$B = \bigcup_{x \in X} (x, B_x),$$

then $\tilde{A} \cap \tilde{B} = 0$ by Proposition 2 and (1). There are disjoint open subsets $G$ and $H$ of $X \times Y$ with $G \supset \tilde{A}$ and $H \supset \tilde{B}$. $\{G[x]; x \in X\}$ and $\{H[x]; x \in X\}$ are the separating families of $\{A_x; x \in X\}$ and $\{B_x; x \in X\}$. In fact, by Corollary 1 to Proposition 2,

$$c\limsup_{a} G[a] \supset A[a] \supset A,$$

$$c\limsup_{a} H[a] \supset B_a$$

for any $a \in X$.

Sufficiency. Suppose that $P(X, Y)$ is satisfied and $A$ and $B$ are disjoint closed subsets of $X \times Y$. Corollary 1 to Proposition 2 follows

$$\limsup_{a} A[a] \cap \limsup_{a} B[a] = 0$$

for any $a \in X$, so there are separating families $\{G_x; x \in X\}$ and $\{H_x; x \in X\}$ of $\{A[x]; x \in X\}$ and $\{B[x]; x \in X\}$. Put

$$G = \bigcup_{x \in X} (x, G_x)$$

and

$$H = \bigcup_{x \in X} (x, H_x),$$

then, by Proposition 2,

$$G^a = c\limsup_{a} G_x \supset A[a]$$

for any $a \in X$, namely, $G^a \supset A$; similarly $H^a \supset B$. Since $G$ and $H$ are disjoint, $G^a$ and $H^a$ separate $A$ and $B$.

Remarks. The following remarks are easily seen from the proof above.

1. If $X \times Y$ satisfies $T_i$ then $P(X, Y)$ and $P(Y, X)$ are both necessary, so by Proposition 4 both $X$ and $Y$ also necessarily satisfy $T_i$. 

(1) If $X \times Y$ satisfies $T_i$ then $P(X, Y)$ and $P(Y, X)$ are both necessary, so by Proposition 4 both $X$ and $Y$ also necessarily satisfy $T_i$. 

(2) In order that $X \times Y$ satisfies $T_\alpha$, one of the properties $P_\alpha(X, Y)$ and $P_\alpha(Y, X)$, say $P_\alpha(X, Y)$, is necessary and sufficient which is given by replacing (i) and (ii) in the definition of $P(X, Y)$ by

\[(i'') \quad c\limsup_a G_x \cap c\limsup_a H_x = \emptyset\]

for any $a \in X$,

\[(ii'') \quad c\limsup_a G_x \supseteq \limsup_a A_x\]

and

\[c\limsup_a H_x \supseteq \limsup_a B_x\]

for any $a \in X$.

Reference