197. *Necessary and Sufficient Conditions for the Normality of the Product of Two Spaces*

By Masahiko ATSUJI
Department of Mathematics, Josai University, Saitama

(Comm. by Kinjirô KUNUGI, M. J. A., Dec. 12, 1969)

In this paper we shall present a solution (Theorem) of the problem to find necessary and sufficient conditions for the normality of a product space $X\times Y$.

This fundamental problem has been researched by many mathematicians probably since about the time (1925) when the importance of normal spaces was found by P. Urysohn [1]. Though the problem has been unsettled for a fairly long time, the proof of our Theorem is simple and elementary. Difficulty may have been in the formulation of the theorem, which is natural but apparently pretty different from ones conjectured from known partial solutions.

In this paper a space is, unless otherwise specified, topological.

Let A be a subset of the product space $X\times Y$ of spaces X and Y, then we write for $x\in X$

$$A[x]=\{y\in Y\,;\,(x,y)\in A\}.$$

Definition 1. Let $\mathfrak{F}$ be a family of neighborhoods of $a\in X$, and let $\{A_x\,;\,x\in Z\subset X\}$ any family of subsets A_x of Y, then we write

$$\limsup_{\mathfrak{F}} A_x=\bigcap_{U\in\mathfrak{F}}\overline{\bigcup_{x\in U} A_x},$$

$$c\text{-}\limsup_{\mathfrak{F}} A_x=Y-\limsup_{\mathfrak{F}}(Y-A_x),$$

where the bar means the closure in Y and $x\in U$ does $x\in U\cap Z$.

Hereafter, let us denote by $\mathfrak{N}_a$ for $a\in X$ the neighborhood system of a in X, and we write "$\limsup_{a}$" instead of "$\limsup_{\mathfrak{N}_a}$". We can easily obtain

Proposition 1. *Let $\mathfrak{M}$ be a neighborhood base of a in X, then*

$$\limsup_{a} A_x=\limsup_{\mathfrak{M}} A_x,$$

$$c\text{-}\limsup_{a} A_x=c\text{-}\limsup_{\mathfrak{M}} A_x.$$

Proposition 2. *Let $\{A_x\,;\,x\in Z\subset X\}$ be any family of sets $A_x\subset Y$, and put*

$$(x,A_x)=\{(x,y)\,;\,y\in A_x\},$$

$$A=\bigcup_{x\in Z}(x,A_x),$$

then

(i) $\bar{A}[a]=\limsup_a A[x]=\limsup_a A_x$,

(ii) $A^0[a]=c\text{-}\limsup_a A[x]=c\text{-}\limsup_a A_x$

for any $a\in X$, *where the bar and* 0 *mean the closure and the interior in* $X\times Y$ respectively.

Proof of (i) is obtained by the following equivalent statements.

$p\in\bar{A}[a]$.

$(a, p)\in\bar{A}$.

$(U\times V)\cap A\neq\emptyset$ for any $U\in\mathfrak{N}_a$ and $V\in\mathfrak{N}_p$.

$V\cap A[x]\neq\emptyset$ for any $U\in\mathfrak{N}_a$, $V\in\mathfrak{N}_p$ and some $x\in U$.

$V\cap\{\bigcup_{x\in U}A[x]\}\neq\emptyset$ for any $U\in\mathfrak{N}_a$ and $V\in\mathfrak{N}_p$.

$p\in\overline{\bigcup_{x\in U}A[x]}$ for any $U\in\mathfrak{N}_a$.

$p\in\bigcap_{U\in\mathfrak{N}_a}\overline{\bigcup_{x\in U}A[x]}$.

The proof of (ii) is obtained by this:

$$\begin{aligned} c\text{-}\limsup_a A[x] &= Y-\limsup_a (Y-A[x]) \\ &= Y-\limsup_a (X\times Y-A)[x] \\ &= \mathcal{C}(\overline{\mathcal{C}A}[a])=(\mathcal{C}\overline{\mathcal{C}A})[a], \end{aligned}$$

where $\mathcal{C}$ means the complement.

Corollary 1. *A set A in* $X\times Y$ *is closed* (*open*) *if and only if*

$$\limsup_a A[x]=A[a]$$

$$(c\text{-}\limsup_a A[x]=A[a])$$

for any $a\in X$.

Corollary 2. *Let* $\{A_x; x\in Z\subset X\}$ *be any family of sets* $A_x\subset Y$, *then*

$$\limsup_a A_x=\limsup_a(\limsup_z A_x),$$

$$c\text{-}\limsup_a A_x=c\text{-}\limsup_a(c\text{-}\limsup_z A_x).$$

Proof. Put

$$A=\bigcup_{x\in X}(x, A_x),$$

$$B=\bigcup_{z\in X}(z, \limsup_z A_x),$$

then

$$\bar{A}[a]=\limsup_a A_x=B[a]$$

for any $a\in X$, so we have

$$\bar{A}=B,$$

$$\limsup_a(\limsup_z A_x)=\bar{B}[a]=B[a]=\limsup_a A_x.$$

Definition 2. The following property is denoted by $P(X, Y)$. Let $\{A_x\subset Y; x\in X\}$ and $\{B_x\subset Y; x\in X\}$ be any families with

$$(*)\qquad \limsup_a A_x\cap\limsup_a B_x=\emptyset$$

for any $a \in X$, then there are families $\{G_x \subset Y; x \in X\}$ and $\{H_x \subset Y; x \in X\}$ satisfying

(i) $$G_x \cap H_x = \emptyset \text{ for any } x \in X,$$

(ii) $$c\text{-}\limsup_a G_x \supset A_a$$

and

$$c\text{-}\limsup_a H_x \supset B_a$$

foy any $a \in X$. $\{G_x; x \in X\}$ and $\{H_x; x \in X\}$ are called the *separating families* (or *separators*) of $\{A_x; x \in X\}$ and $\{B_x; x \in X\}$.

Remark. We consider A_x and B_x are defined for every point x of X, and they may be empty for some x. Considering Corollary 2 above, in Definition 2 we can assume that A_x and B_x are closed and G_x and H_x are open. It is convenient to have another formulation: $c\text{-}\limsup_a A_x = \bigcup_{U \in \mathfrak{N}_a} (\bigcap_{x \in U} A_x)^0$.

Proposition 3. *The property $P(X, Y)$ is equivalent with the following property $P_1(X, Y)$. Let $\{A_x \subset Y; x \in X\}$ and $\{B_x \subset Y; x \in X\}$ be any families with*

(*) $$\limsup_a A_x \cap \limsup_a B_x = \emptyset$$

for any $a \in X$, then there is a family $\{G_x \subset Y; x \in X\}$ satisfying

(i) $$c\text{-}\limsup_a G_x \supset A_a$$

and

(ii) $$\limsup_a G_x \cap B_a = \emptyset$$

for any $a \in X$.

Proof. Suppose that $P(X, Y)$ is satisfied, then there are $\{G_x; x \in X\}$ and $\{H_x; x \in X\}$ satisfying (i) and (ii) in Definition 2, and we have

$$B_a \subset c\text{-}\limsup_a H_x \subset c\text{-}\limsup_a \mathcal{C}G_x = \mathcal{C}(\limsup_a G_x),$$

namely,

$$\limsup_a G_x \cap B_a = \emptyset.$$

Assuming conversely $P_1(X, Y)$, we put $H_x = \mathcal{C}G_x$, then

$$B_a \subset \mathcal{C}(\limsup_a G_x) = c\text{-}\limsup_a H_x.$$

It is said that a space satisfies the *separation axiom T_4* if any two disjoint closed subsets of the space are separated by two disjoint open subsets.

Proposition 4. *If $P(X, Y)$ is satisfied, then Y satisfies the separation axiom T_4*

Proof. Let A and B be any two disjoint closed subsets of Y. Take a point $c \in X$, and put $A_c = A$ and $B_c = B$, and $A_x = B_x = \emptyset$ for $x \neq c$, then

$$\limsup_a A_x \cap \limsup_a B_x = \emptyset$$

for any $a \in X$, so there are separating families $\{G_x \subset Y; x \in X\}$ and $\{H_x \subset Y; x \in X\}$ of $\{A_x\}$ and $\{B_x\}$. T_4 follows from

$$G_c^0 \supset c\text{-}\limsup_c G_x \supset A_c,$$

$$H_c^0 \supset c\text{-}\limsup_c H_x \supset B_c.$$

Now we can prove our main theorem.

Theorem. *One of the properties $P(X, Y)$ and $P(Y, X)$ is necessary and sufficient in order that the product space $X \times Y$ satisfies the separation axiom T_4.*

Proof. *Necessity.* Suppose that $X \times Y$ satisfies T_4 and that $\{A_x \subset Y; x \in X\}$ and $\{B_x \subset Y; x \in X\}$ fulfil

(1) $$\limsup_a A_x \cap \limsup_a B_x = \emptyset$$

for any $a \in X$. Put

$$A = \bigcup_{x \in X} (x, A_x)$$

and

$$B = \bigcup_{x \in X} (x, B_x),$$

then $\bar{A} \cap \bar{B} = \emptyset$ by Proposition 2 and (1). There are disjoint open subsets G and H of $X \times Y$ with $G \supset \bar{A}$ and $H \supset \bar{B}$. $\{G[x]; x \in X\}$ and $\{H[x]; x \in X\}$ are the separating families of $\{A_x; x \in X\}$ and $\{B_x; x \in X\}$. In fact, by Corollary 1 to Proposition 2,

$$c\text{-}\limsup_a G[x] = G[a] \supset \bar{A}[a] \supset A_a,$$

$$c\text{-}\limsup_a H[x] \supset B_a$$

for any $a \in X$.

Sufficiency. Suppose that $P(X, Y)$ is satisfied and A and B are disjoint closed subsets of $X \times Y$. Corollary 1 to Proposition 2 follows

$$\limsup_a A[x] \cap \limsup_a B[x] = \emptyset$$

for any $a \in X$, so there are separating families $\{G_x; x \in X\}$ and $\{H_x; x \in X\}$ of $\{A[x]; x \in X\}$ and $\{B[x]; x \in X\}$. Put

$$G = \bigcup_{x \in X} (x, G_x)$$

and

$$H = \bigcup_{x \in X} (x, H_x),$$

then, by Proposition 2,

$$G^0[a] = c\text{-}\limsup_a G_x \supset A[a]$$

for any $a \in X$, namely, $G^0 \supset A$; similarly $H^0 \supset B$. Since G and H are disjoint, G^0 and H^0 separate A and B.

Remarks. The following remarks are easily seen from the proof above.

(1) If $X \times Y$ satisfies T_4 then $P(X, Y)$ and $P(Y, X)$ are both necessary, so by Proposition 4 both X and Y also necessarily satisfy T_4.

(2) In order that $X \times Y$ satisfies T_4, one of the properties $P_2(X, Y)$ and $P_2(Y, X)$, say $P_2(X, Y)$, is necessary and sufficient which is given by replacing (i) and (ii) in the definition of $P(X, Y)$ by

(i″) $$c\text{-}\limsup_{a} G_x \cap c\text{-}\limsup_{a} H_x = \emptyset$$

for any $a \in X$,

(ii″) $$c\text{-}\limsup_{a} G_x \supset \limsup_{a} A_x$$

and

$$c\text{-}\limsup_{a} H_x \supset \limsup_{a} B_x$$

for any $a \in X$.

Reference

[1] P. Urysohn: Ueber die Möchtigkeit der zusammenhängenden Mengen. Math. Ann., **94**, 262–295 (1925).