196. *Z-mappings and C*-embeddings*

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Recently, Comfort and Negrepontis investigated the interesting properties of proper C*-pair [1]. In this paper, we shall give in § 1 a necessary and sufficient condition that $X\times Y$ is C^*-embedded in $X\times\beta Y$ and give in § 2 partial answers to the problems with respect to the product spaces raised by Morita.

Throughout this paper, we assume that our spaces are completely regular T_1-spaces and mappings are continuous. We will use the same notations as in [3]; for instance, the symbol βX denotes the Stone Čech compactification of a given space X. We denote by λ the projection: $X\times Y\to X$ and put $W=X\times\beta Y$.

§ 1. Relations between *Z*-mappings and *C-embeddings.**

We call a mapping φ from X onto Y a *Z-mapping* if φE is closed in Y for every zero set E of X. A closed mapping is always a Z-mapping ([5], 1.1).

1.1. Theorem. *$X\times Y$ is C^*-embedded in $X\times\beta Y$ if and only if the projection $\lambda: X\times Y\to X$ is a Z-mapping.*

Proof. *Necessity.* Let F be a zero set of $X\times Y$; that is, there is a function $f\in C^*(X\times Y)$ such that $F=\{(x,y); f(x,y)=0\}$ and $0\le f\le 1$ on $X\times Y$. Now suppose that there exists a point $x_0\in \mathrm{cl}\,\lambda F-\lambda F$. Since βY is compact, the projection $\pi: W=X\times\beta Y\to X$ is closed. $\mathrm{Cl}_W F$ being closed, $\pi(\mathrm{cl}_W F)$ contains x_0. On the other hand, $x_0\notin F$ implies that f is positive on $\{x_0\}\times Y$. We shall consider the function g defined in the following way:

$$g(x,y)=(f\,|\,(\{x_0\}\times Y))(x_0,y)\quad\text{for}\quad (x,y)\in X\times Y.$$

It is easy to see that g is continuous and $0\le g\le 1$. Define

$$h(x,y)=(f(x,y)/g(x,y))\wedge 1.$$

The function h is continuous and $F=Z(h)$ and $h=1$ on $\{x_0\}\times Y$. We denote by k the continuous extension of h over $X\times\beta Y$. Obviously $k=1$ on $\{x_0\}\times\beta Y$. This shows that $\mathrm{cl}_W F\cap\{x_0\}\times\beta Y=\emptyset$ which is impossible.

Sufficiency. In Theorem 3.1 in [1], it is proved that if λ is closed, then $X\times Y$ is C^*-embedded in $X\times\beta Y$. In its proof, it is easy to check that "closedness of λ" is replaced by "λ being a Z-mapping".

Remark. $X\times Y$ is not necessarily C^*-embedded in $\beta X\times\beta Y$ even if both projections: $X\times Y\to X$ and $X\times Y\to Y$ are Z-mappings (for instance, both spaces X and Y are discrete [1], [4]).

As application of Theorem 1.1 we have

1.2. Corollary. *If λ is a Z-mapping, then the projection:* $X\times Z \to X$ *is always a Z-mapping for any subspace Z of* βY *containing Y.*

This follows from Theorem 1.1 and the fact that if $X\times Y$ is C^*-embedded in $X\times \beta Y$, then so is $X\times Z$.

It is known that if X is a P-space and Y is Lindelöf space, then λ is closed (Theorem 3.2 in [1] or see the proof of Lemma 8.1 in [5]). From this fact and 1.2 we have

1.3. Corollary. *If X is a P-space and Z is a subspace of* βY *containing Y where Y is a Lindelöf space, then* $X\times Z$ *is* C^**-embedded in* $X\times \beta Y$.

In Theorem 2.1 in [5] we have proved that X is pseudocompact if and only if the projection: $X\times Y\to Y$ is a Z-mapping for any weakly separable space Y. From this fact and 1.2 we have

1.4. Corollary. *X is pseudocompact if and only if* $X\times Y$ *is* C^**-embedded in* $\beta X\times Y$ *for any weakly separable space Y.*

In Theorem 5.3 in [1] it is proved that if the cardinal number of Y is nonmeasurable and $X\times Y$ is C^*-embedded in $X\times \beta Y$, then $\upsilon(X\times Y) = \upsilon X\times \upsilon Y$ where υX is the Hewitt realcompactification of X. From this fact and 1.2 we have

1.5. Corollary. *If the cardinal number of Y is nonmeasurable and the projection:* $X\times Y\to X$ *is a Z-mapping, then* $\upsilon(X\times Z)=\upsilon X\times \upsilon Z$ *for any subspace Z of* βY *containing Y.*

Tamano [7] has proved that if both X and Y are pseudocompact, then $\beta(X\times Y)=\beta X\times \beta Y$ if and only if λ is a Z-mapping. From this fact we have

1.6. Corollary. *If both spaces X and Y are pseudocompact, then* C^**-embedding of* $X\times Y$ *in* $X\times \beta Y$ *implies* C^**-embedding of* $X\times Y$ *in* $\beta X\times \beta Y$.

§2. Partial answers to Morita Problems.

The followings were raised by Morita.

(M1) *Let Y' be an image of a normal space Y under a closed mapping and let X be a compact space. Is* $X\times Y'$ *normal whenever* $X\times Y$ *is normal?*

(M2) *Let Y be a metric space. Is* $X\times Y$ *countably paracompact whenever* $X\times Y$ *is normal?*

We shall consider the above problems under the assumption that $X\times Y$ is C^*-embedded in $X\times \beta Y$. The following theorem 2.3 (cf. 2.4) and 2.5 are affirmative answeres to (M1) and (M2) respectively under the above condition.

Before considering the problems, we shall prove the following theorem which asserts that *if X is an image of a weakly separable* *space under a closed mapping, then* $X\times Y$ *is* C^**-embedded in* $X\times\beta Y$ *for any pseudocompact space* Y *(see Corollary* 1.4) *and moreover* $\upsilon(X\times Y)=\upsilon X\times\upsilon Y$ *holds under the assumption that the cardinal number of* Y *is nonmeasurable* (see Corollary 1.5).

2.1. Theorem. *Let* φ_1 *be a closed mapping from* X *onto* X' *and let* φ_2 *be a continuous mapping from* Y *onto* Y'. *If* $X\times Y$ *is* C^**-embedded in* $X\times\beta Y$*, then* $X'\times Y'$ *is* C^**-embedded in* $X'\times\beta Y'$.

Proof. It is sufficient by Theorem 1.1 to show that the projection $\mu: X'\times Y'\to X'$ is a Z-mapping. Let $E=Z(f)$ be a zero set of $X'\times Y'$. Define the mapping $\varphi: \varphi(x,y)=(\varphi_1(x),\varphi_2(y))$ from $X\times Y$ onto $X'\times Y'$ and put $g==f\varphi$. It is easy to see that $F=Z(g)=\varphi^{-1}(E)$. F being a zero set of $X\times Y$ and λ being a Z-mapping by Theorem 1.1, λF is closed in X and hence so is $\varphi_1\lambda F$. We shall show that $\varphi_1\lambda F=\mu E$ which completes the proof. $x\in\lambda F\leftrightarrow(x,y)\in F$ for some point $y\in Y$ $\leftrightarrow(x,y)\in\varphi^{-1}(E)\leftrightarrow\varphi(x,y)\in E\leftrightarrow(\varphi_1x,\varphi_2y)\in E$. Thus $x\in\lambda F$ implies that $\varphi_1x\in\mu E$, and hence $\varphi_1\lambda F\subset\mu E$. Conversely $x_1\in\mu E\leftrightarrow(x_1,y_1)\in E$ for some point $y_1\in Y'$. Since $F=\varphi^{-1}(E)$, there exists a point $(x,y)\in F$ with $\varphi_1x=x_1$ and $\varphi_2y=y_1$. This measns that $x\in\lambda F$ and $x_1\in\varphi_1\lambda F$, that is, $\mu E\subset\varphi_1\lambda F$.

The following lemma asserts that under suitable conditions, the closure in $X\times\beta Y$ of closed subsets of $X\times Y$ may be computed by taking closures of vertical slices.

2.2. Lemma. *Let* $X\times Y$ *be normal and* C^**-embedded in* $X\times\beta Y$ *and let* E *be a closed subset of* $X\times Y$. *If* $z\in\mathrm{cl}_W E-E$ *and* $\pi(z)=x_0$, *then* $z\in\mathrm{cl}_W(E\cap Y_0)$ *where* π *is the projection:* $W=X\times\beta Y\to X$ *and* $Y_0=\{x_0\}\times Y$.

Proof. Let us put $E_0=E\cap Y_0$. If $E_0=\emptyset$, then $X\times Y$ being normal, there is a function $f\in C^*(X\times Y)$ such that $f=0$ on Y_0 and $f=1$ on E. Since $X\times Y$ is C^*-embedded in $X\times\beta Y$, it is easily seen that $\mathrm{cl}_W E\cap\mathrm{cl}_W Y_0=\emptyset$. But this is a contradiction because $z\in\mathrm{cl}_W E$ and $\mathrm{cl}_W Y_0=\{x_0\}\times\beta Y$.

Now suppose that $z\notin\mathrm{cl}_W E_0(\neq\emptyset)$. There exists a neighborhood U (in W) of z such that $\mathrm{cl}_W U\cap\mathrm{cl}_W E_0=\emptyset$. Let us put $\mathrm{cl}_W U\cap E=E_1$ and $\mathrm{cl}_W U\cap Y_0=A$. Then $E_1\cap A=\emptyset$. As similar to the above method, we have contradistinctive relations: $\mathrm{cl}_W E_1\cap\mathrm{cl}_W A=\emptyset$ and $z\in\mathrm{cl}_W E_1\cap\mathrm{cl}_W A$.

2.3. Theorem. *Let* φ_1 *be a perfect mapping from* X *onto* X', φ_2 *a closed mapping from* Y *onto* Y' *and let* $X\times Y$ *be* C^**-embedded in* $X\times\beta Y$. *If* $X\times Y$ *is normal, then so is* $X'\times Y'$.

Proof. Let E_1 and E_2 be disjoint closed subsets of $X'\times Y'$. Define

$$\varphi(x,y)=(\varphi_1x,\varphi_2y)\quad\text{and}\quad F_i=\varphi^{-1}E_i(i=1,2).$$

Since $X\times Y$ is normal and $F_1\cap F_2=\emptyset$, there exists a function f such that $F_1 \subset Z(f)$ and $F_2 \subset \{(x, y); f(x, y)=1\}$. By the assumption, f has the continuous extension g over $W=X\times\beta Y$. Obviously $\mathrm{cl}_W F_1 = B_1 \subset Z(g)$ and $\mathrm{cl}_W F_2 = B_2 \subset \{(x, y); g(x, y)=1\}$. Let ψ_2 be the Stone extension of φ_2 from βY onto $\beta Y'$. The mapping ψ from W onto $W'=X'\times\beta Y'$ defined by

$$\psi(x, y)=(\varphi_1 x, \psi_2 y) \quad \text{for} \quad (x, y)\in W$$

is perfect by [6] because both mappings φ_1 and ψ_2 are perfect.

Now suppose that $(x', y')\in\psi(B_1)\cap\psi(B_2)$. There exists a point $(x_i, y_i)\in B_i$ with $\psi(x_i, y_i)=(x', y')$ $(i=1,2)$. It is obvious that the point (x_i, y_i) must be contained in $X\times(\beta Y - Y)$ $(i=1,2)$. Let us put $A_i = F_i \cap Y_i$ where $Y_i=\{x_i\}\times Y$. By virture of Lemma 2.2 we have $(x_i, y_i)\in \mathrm{cl}_W A_i$ which means that

$$(x', y')=\psi(x_1, y_1)\in\psi(\mathrm{cl}_W A_i)\subset \mathrm{cl}_{W'}(\psi A_i)=\mathrm{cl}_{W'}\varphi A_i.$$

On the other hand $\varphi A_i \subset E_i\cap(\{x'\}\times Y')$ and $\varphi A_1\cap\varphi A_2=\emptyset$. φ_2 being closed and Y being normal, Y' is normal and so is $\{x'\}\times Y'$. $\{x'\}\times Y'$ is C^*-embedded in $X'\times Y'$ and so is $X'\times Y'$ in $X'\times\beta Y'$ by Theorem 2.1. Thus $\{x'\}\times Y'$ is C^*-embedded in $X'\times\beta Y'$ and $\mathrm{cl}_{W'}(\{x'\}\times Y')=\{x'\}\times\beta Y' =\beta(\{x'\}\times Y')$. φA_1 and φA_1 are closed disjoint subsets of $\{x'\}\times Y'$ and hence $\mathrm{cl}_{W'}\varphi A_1\cap \mathrm{cl}_{W'}\varphi A_2=\emptyset$. This is impossible because $(x', y')\in \mathrm{cl}_{W'}A_i$ $(i=1,2)$. Thus we have that $\psi B_1\cap\psi B_2=\emptyset$.

In W, we put $U_1=\{(x, y); g(x, y)<1/3\}$ and $U_2=\{(x, y); g(x, y) >2/3\}$. $B_i\subset U_i$ $(i=1,2)$. Since ψ is closed, $V_i=W'-\psi(W-U_i)$ is open and obviously $V_1\cap V_2=\emptyset$. Next we shall prove that $E_i\subset V_i$. Suppose that there exists a point $(x_0, y_0)\in W-U_1$ and $\psi(x_0, y_0)=(x_1, y_1) \in E_1$. Let G be a closed neighborhood (in W) of (x_0, y_0) which is disjoint from B_1 and let us put $K=G\cap(\{x_0\}\times Y)$ and $H=F_1\cap(\{x_0\}\times Y)$. Since $X\times Y$ is dense in W, we have $(x_0, y_0)\in \mathrm{cl}_W K$ by Lemma 2.2. $\varphi(x_0, \varphi_2^{-1}(y_0))=(x_1, y_1)$ implies $H\neq\emptyset$. Since φ_2 be considered as a closed mapping from $\{x_0\}\times Y$ onto $\{x_1\}\times Y'$ and $(x_1, y_1)\in\{x_1\}\times Y'$, we have that $(x_1, y_1)\in\varphi_2(K)\cap\varphi_2(H)$. On the other hand, $\varphi^{-1}(E_1)=F_1$ and $F_1\cap G=\emptyset$ which implies that $\varphi_2(K)\cap\varphi_2(H)=\emptyset$ and hence $E_1\subset V_1$. Similarly we have that $E_2\subset V_2$.

The argument above leads that E_1 and E_2 are separated by disjoint open subsets $V_1\cap(X'\times Y')$ and $V_2\cap(X'\times Y')$, that is, $X'\times Y'$ must be normal.

In Theorem 2.1 in [1], it is proved that if $X\times Y$ is C^*-embedded in $X\times\beta Y$, then either X is a P-space or Y is pseudocompact. Thus we have that if in Theorem 3.3 X is compact the C^*-embedding of $X\times Y$ in $X\times\beta Y$ implies pseudocompactness of Y (and hence Y must be countably compact). On the other hand if X is compact and Y is countably compact then $\beta(X\times Y)=\beta X\times\beta Y$ [4]. Thus we have

2.4. Theorem. *Let Y' be an image of a countably compact* *space Y under a closed mapping and let X' be an image of a compact space X under a continuous mapping. If $X\times Y$ is normal, then so is $X'\times Y'$.*

Dowker [2] has proved that a normal space X is countably paracompact if and only if $X\times[0,1]$ is normal. By the analogous method we can prove the following: *Let Y be a space having the property (∗) there exists a countable discrete subset A with $\mathrm{cl}_Y A\neq A$, then the normality of $X\times Y$ implies the countable paracompactness of X.* Using this fact we shall prove the following

2.5. *Let Y be a countably compact space. If $X\times Y$ is normal and C^*-embedded in $X\times\beta Y$, then $X\times Y$ is countably paracompact.*

Proof. Let $\{F_n\}$ be a decreasing sequence of closed sets of $X\times Y$ with $\cap F_n=\emptyset$ and let $F_{n,x}=(\{x\}\times Y)\cap F_n$ for any point $x\in X$. Since $\cap F_{n,x}=\emptyset$ and Y is countably compact, there exists an integer $n=n(x)$ with $F_{n,x}=\emptyset$ for each point $x\in X$. This implies that $\cap\lambda F_n=\emptyset$. Since countably compact spaces have the property (∗), X is countably paracompact by the remark above. On the other hand λ is a Z-mapping and moreover a closed mapping ([5], 1.3) because $X\times Y$ is normal. Thus λF_n is closed and hence by virture of countable paracompactness of X, there are open sets 0_n for each n such that $\lambda F_n\subset 0_n$ and $\cap\bar{0}_n=\emptyset$. Thus $\cap(\bar{0}_n\times Y)=\emptyset$ and $0_n\times Y\supset F_n$ which completes the proof.

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