194. *On a Set Theory Suggested by Dedecker and Ehresmann. I*[*]

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1. Introduction. As it is known, the common systems of set theory do not provide an adequate foundation for the theory of categories (see, for example [2] and the references given there). Various solutions have been proposed to surmount this difficulty (cf. [2]-[3], [7], [8] and [14]). Using an idea of Ehresmann [6], Dedecker informally describes in [5] a set theory reputedly appropriate to serve as a basis for category theory. The object of the present paper is to formalize Dedecker's system (or, more precisely, to describe a formal system belonging to the type suggested by Dedecker and Ehresmann).

We were led to the formalization of dedecker's system, called here system $\boldsymbol{D}$, studying questions of a very different nature (cf. [4]). In fact, $\boldsymbol{D}$ is the first of a hierarchy of set theories (similar to the hierarchy defined in [1]) which will be studied in the near future.

In systems of the Von Neumann-Bernays-Gödel type, like the Kelley-Morse set theory ([9], appendix), a distinction is made between sets and classes and one is able to operate on sets with the classical rules, but the same is in general not true of the operations with classes. In $\boldsymbol{D}$, on the contrary, it is possible to *operate* on classes (proper or not) as one does with sets; for instance, the class of equivalence classes of a given class, corresponding to an equivalence relation always exists, and the unit class of any class is in all cases defined, such notions having the desired suitable properties.

In a few words, $\boldsymbol{D}$ is a combination of the Kelley-Morse set theory with the Quine-Rosser NF system ([12] and [13]).

(We presuppose that the reader has a good knowledge of [5], [9] and [13].)

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2. The system D. $A, B, C, \cdots$ and $\alpha, \beta, \gamma, \cdots$ are used as (intuitive) metalinguistic variables.

2.1. Formal symbols. The primitive formal symbols of D are:

2.1.1. Logical symbols: $\vee$ (or), $\neg$ (not), $\forall$ (for all).

2.1.2. Predicate symbols: $=$ (equals), $\in$ (belongs).

2.1.3. Variables: $t, u, v, x, y, z, t', u', v', x', y', z', t'', \cdots$

2.1.4. Parentheses: (,).

2.1.5. One individual symbol: V.

2.1.6. Classifiers: $\{ : \}$, $\{ : \}_V$.

Remark. The abbreviations $\supset$ (implies), & (and), $\equiv$ (equivalent) and $\exists$ (there exists) will be used as it is customary.

2.2. Terms and formulas of *D*.

2.2.1. If α and β are terms, then $\alpha=\beta$ and $\alpha \in \beta$ are formulas.

2.2.2. If A and B are formulas, then $(A)\vee(B)$ and $\neg(A)$ are formulas.

2.2.3. If A is a formula and α is a variable, $\forall\alpha(A)$ is a formula.

2.2.4. The variables and the individual symbol are terms.

2.2.5. If α is a variable and F is a formula, $\{\alpha : F\}$ and $\{\alpha : F\}_V$ are terms.

2.2.6. The only terms and formulas are those given by 2.2.1.–2.2.5.

Remark. The notions of free variable, bound variable, term free for a variable in a formula, etc., are defined as usual; the conventions of [10] are used without explicit mention. It is also easy to define the concepts of *stratified formula* and *stratified term* (cf. [13]). It is convenient to note that a stratified formula or term is allowed to contain V and we may allow that different occurrences of V in a formula or term can have different subscripts attached for the purpose of stratification.

2.3. Logical postulates for *D*.

2.3.1. *Postulates for the propositional calculus.*

The symbols A, B, C denote formulas:

I_1: $A \vee A \supset A$.

I_2: $A \supset A \vee B$.

I_3: $A \vee B \supset B \vee A$.

I_4: $(A \supset B) \supset (C \vee A \supset C \vee B)$.

I_5: $\dfrac{A \quad A \supset B}{B}$

2.3.2. *Postulates for the predicate calculus.*

α is a variable, $A(\alpha)$ is a formula and C is a formura which does not contain α free:

$$II_1:\ \frac{C \supset A(\alpha)}{C \supset \forall \alpha A(\alpha)}.$$

$A(\alpha)$ is a formula and β is a term free for the variable α in $A(\alpha)$:

II_2: $\forall\alpha A(\alpha)\supset A(\beta)$.

2.3.3. *Postulates for equality.*

III_1: $\forall x(x=x)$.

$A(\alpha)$ is a formula, α is a variable and β and γ are distinct variables free for α in $A(\alpha)$:

III_2: $\beta=\gamma\supset(A(\beta)\equiv A(\gamma))$.

Remark. The concepts of (formal) proof, (formal deduction), etc., are defined as in [10].

2.4. Specific postulates of $\boldsymbol{D}$.

In the remainder of this section, we shall proceed as follows. The specific postulates of $\boldsymbol{D}$ will be introduced and, at the same time, some very simple theorems and definitions will be given. These results will be sufficient to give a reasonable idea of $\boldsymbol{D}$.

Definition 1. If α and β are terms and γ is a variable not occurring in α and β, then:

$$\alpha\subset\beta\equiv\forall\gamma(\gamma\in\alpha\supset\gamma\in\beta).$$

Definition 2. *If α and β are terms,*

$$\alpha\notin\beta\equiv\neg(\alpha\in\beta),$$
$$\alpha\neq\beta\equiv\neg(\alpha=\beta).$$

Postulate of extent.

(P1) $$\forall z(z\in x\equiv z\in y)\supset x=y.$$

Postulates of classification. α and β are variables, $F(\alpha)$ is a stratified formula, β is free for α in $F(\alpha)$ and does not occur free in $F(\alpha)$:

(P2) $$\beta\in\{\alpha: F(\alpha)\}\equiv F(\beta).$$

(Note: If we say that a formula containning $\boldsymbol{V}$ is not stratified, we *can still use it to determine* a class.)

$F(\alpha)$ is a formula, α and β are variables, β is free for α in $F(\alpha)$ and does not appear free in $F(\alpha)$:

(P3) $$\beta\in\{\alpha: F(\alpha)\}_V\equiv\beta\in\boldsymbol{V}\ \&\ F(\beta).$$

Theorem 1. $\vdash\forall z(z\in x\equiv z\in y)\equiv x=y$.

Theorem 2. $\vdash x\subset x$;

$\vdash x\subset y\ \&\ y\subset x\supset x=y$;

$\vdash x\subset y\ \&\ y\subset z\supset x\subset z$.

Theorem 3. $\vdash x\subset y\ \&\ y\subset x\equiv x=y$.

Postulate of normality.

(P4) $$\forall x(x\in\boldsymbol{V}\supset x\subset\boldsymbol{V}).$$

Definition 3. $\boldsymbol{U}=\{x: x=x\}$, $\boldsymbol{C}=\{x: x\subset\boldsymbol{V}\}$, $\emptyset=\{x: x\neq x\}$.

Theorem 4. $\vdash\forall x(x\in\boldsymbol{U})$;

$\vdash\boldsymbol{U}\in\boldsymbol{U}$;

$\vdash\boldsymbol{V}\in\boldsymbol{C}$;

$\vdash\boldsymbol{V}\in\boldsymbol{U}$;

$\vdash \boldsymbol{C} \in \boldsymbol{U}$;
$\vdash \boldsymbol{V} \subset \boldsymbol{C}$;
$\vdash \boldsymbol{C} \subset \boldsymbol{U}$;
$\vdash \forall x(x \notin \emptyset)$;
$\vdash \exists x(x \in \boldsymbol{C} \ \& \ x \notin \boldsymbol{V})$;
$\vdash \exists x(x \in \boldsymbol{U} \ \& \ x \notin \boldsymbol{C})$.

Definition 4. If α is a term, then:

$$\alpha \text{ is a class} \equiv \alpha \in \boldsymbol{U},$$
$$\alpha \text{ is a set} \equiv \alpha \in \boldsymbol{V},$$
$$\alpha \text{ is an elementary class} \equiv \alpha \subset \boldsymbol{V},$$
$$\alpha \text{ is an elementary proper class} \equiv \alpha \subset \boldsymbol{V} \ \& \ \alpha \notin \boldsymbol{V},$$
$$\alpha \text{ is a proper class} \equiv \alpha \notin \boldsymbol{V}.$$

Definition 5. If α and β are terms, then:

$$\alpha \text{ is a subclass of } \beta \equiv \alpha \subset \beta,$$
$$\alpha \text{ is a subset of } \beta \equiv \alpha \subset \beta \ \& \ \alpha \notin \boldsymbol{V}.$$

Theorem 5. $\forall x \forall y(x \in y \ \& \ y \textit{ is a set} \supset x \textit{ is a set})$.

The postulates already stated allow us to *operate* suitably on classes (see [13]). In particular, for an arbitrary class, there exists a class of all its subclasses and, given an equivalence relation on an arbitrary class, one can construct the class of its equivalence classes; and these notions have the expected properties.

As Specker showed for NF ([15]), in $\boldsymbol{D}$ the axiom of choice (in its general form) is not true and it is possible to prove the axiom of infinity for classes (but apparently not for sets).

Definition 6. If α and β are terms and γ is a variable distinct from the variables of α and β, then:

$$\alpha \cup \beta = \{\gamma : \gamma \in \alpha \vee \gamma \in \beta\},$$
$$\alpha \cap \beta = \{\gamma : \gamma \in \alpha \ \& \ \gamma \in \beta\}.$$

Theorem 6. $\vdash x \cup y = y \cup x$;
$\vdash x \cap y = y \cap x$;
$\vdash x \cup x = x$;
$\vdash x \cap x = x$;
$\vdash x \cup (x \cap y) = x$;
$\vdash x \cap (x \cup y) = x$;
$\vdash (x \cup y) \cup z = x \cup (y \cup z)$;
$\vdash (x \cap y) \cap z = x \cap (y \cap z)$;
$\vdash x \cup (y \cap z) = (x \cup y) \cap (x \cup z)$;
$\vdash x \cap (y \cup z) = (x \cap y) \cup (x \cap z)$;
$\vdash x \cup \boldsymbol{U} = \boldsymbol{U}$;
$\vdash x \cap \boldsymbol{U} = \boldsymbol{U}$;
$\vdash x \cup \emptyset = x$;
$\vdash x \cap \emptyset = \emptyset$.

Definition 7. If α is a term and β is a variable not occurring in α, then:

$$\sim\alpha=\{\beta : \beta \notin \alpha\}.$$

Theorem 7. $\vdash \sim\sim x = x$;

$\vdash \sim(x \cup y) = \sim x \cap \sim y$;

$\vdash \sim(x \cap y) = \sim x \cup \sim y$;

$\vdash \sim \boldsymbol{U} = \emptyset$;

$\vdash \sim \emptyset = \boldsymbol{U}$;

$\vdash x \cup \sim x = \boldsymbol{U}$;

$\vdash x \cap \sim x = \emptyset$.

Definition 8. α and β are terms and γ is a variable distinct from the variables of α and β:

$$\alpha \cup_{\boldsymbol{V}} \beta = \{\gamma : \gamma \in \alpha \vee \gamma \in \beta\}_{\boldsymbol{V}},$$
$$\alpha \cap_{\boldsymbol{V}} \beta = \{\gamma : \gamma \in \alpha \;\&\; \gamma \in \beta\}_{\boldsymbol{V}},$$
$$\sim_{\boldsymbol{V}} \alpha = \{\gamma : \gamma \notin \alpha\}_{\boldsymbol{V}}.$$

(Similarly, we may *restrict* to $\boldsymbol{V}$ all definitions of Kelley-Morse set theory [9]).

Theorem 8. $\vdash x \in \boldsymbol{V} \;\&\; y \in \boldsymbol{V} \supset x \cup_{\boldsymbol{V}} y = x \cup y$;

$\vdash x \in \boldsymbol{V} \;\&\; y \in \boldsymbol{V} \supset x \cap_{\boldsymbol{V}} y = x \cap y$;

$\vdash x \in \boldsymbol{V} \supset \sim_{\boldsymbol{V}} x = \sim x \cap \boldsymbol{V}$;

$\vdash (x \cap y) \cap \boldsymbol{V} = x \cap_{\boldsymbol{V}} y$;

$\vdash (x \cup y) \cap \boldsymbol{V} = x \cup_{\boldsymbol{V}} y$.