186. *Realization of Irreducible Bounded Symmetric Domain of Type* (VI)

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1. This is a continuation of our preceding note [3] which appeared in these Proceedings. We shall present here, without proof, the *canonical bounded model* of the irreducible bounded symmetric domain of exceptional type (VI) in the sense of [4].

As was pointed out in [4], we need at first to describe explicitly the irreducible representation of the complex simple Lie algebra of type E_7 which is of the lowest degree, 56. Such a representation was previously discussed by several authors, for instance by H. Freudenthal; however a presentation of that representation which suited our purpose was recently given by R. B. Brown [1] for the first time. His result will be, therefore, briefly reproduced in the following sections 2-3. As for the notation we refer the reader to [3], [4].

2. Let $\mathfrak{J}$ denote the exceptional simple Jordan algebra as described in [1]-[3]; namely $\mathfrak{J}$ is the totality of the (3.3)-hermitian matrices over the complex Cayley numbers $\mathfrak{C}$. The canonical non-degenerate inner-product (u, v) in $\mathfrak{J}$ will be introduced by $(u, v)=\text{Trace}(u\circ v)$, $(u, v\in\mathfrak{J})$ (cf. [1], [2], [5]), for which we consider the dual $\mathfrak{J}^*$ of $\mathfrak{J}$ and will identify hereafter $\mathfrak{J}^*$ with $\mathfrak{J}$ through this inner-product. Now we introduce a 56-dimensional complex vector space V by putting

(1) $$V=V_1\oplus V_2\oplus V_3\oplus V_4,$$

where both V_1 and V_4 are of 1-dimension and $V_2=\mathfrak{J}^*$, $V_3=\mathfrak{J}$. The element x of V is then written as

(2) $$x=\alpha f_1+u^*+v+\beta f_2;\ \alpha, \beta\in C,\ u, v\in\mathfrak{J},$$

where f_1, f_2 denote, respectively, the generators of V_1, V_4 and $u^*\in\mathfrak{J}^*$ is defined by $u^*(v)=(u, v)$ for all $v\in\mathfrak{J}$. After R. B. Brown we introduce in V a non-associative algebra structure $\mathfrak{B}$ by the following rule:

i) $f_if_i=f_1$ $(i=1, 2)$, $\quad f_1f_2=f_2f_1=0$

ii) $f_1u=\frac{1}{3}u$, $\quad f_2u=\frac{2}{3}$; $\quad f_1v^*=\frac{2}{3}v^*$, $\quad f_2v^*=\frac{1}{3}v^*$

iii) $uf_1=0$, $\quad uf_2=u$; $\quad v^*f_1=v^*$, $\quad v^*f_2=0$

iv) $uv^*=(u, v)f_1$, $\quad u^*v=(u, v)f_2$

v) $uv=2(u\times v)^*$, $\quad u^*v^*=2(u\times v)$

$(u, v\in\mathfrak{J})$, where the crossed product $u\times v$ in $\mathfrak{J}$ is given through $(u\times v, w)=3(u, v, w)$ (for $w\in\mathfrak{J}$), the right hand side being the tri-linear form on $\mathfrak{J}$ obtained by linearizing the cubic from on $\mathfrak{J}$ (see, [1], [5]):

$$\det(u)=\xi_1\xi_2\xi_3+2(x_3x_1, \bar{x}_2)-\sum_{i=1}^{3}\xi_i x_i x_i \quad (u \in \mathfrak{J}).$$

3. In the algebra $\mathfrak{V}$ thus introduced we define the trace-function and the non-degenerate inner-product as follows:

$$\mathrm{Trace}\,(x)=\alpha+\beta, \quad (x, y)=\mathrm{Trace}\,(xy)$$

Then, if we write $x=\alpha f_1+a^*+b+\beta f_2$, $y=\xi f_1+c^*+d+\eta f_2$ $(\alpha, \beta, \xi, \eta \in C;\ a, b, c, d \in \mathfrak{J})$, we get

$$(x, y)=\alpha\xi+\beta\eta+(a, d)+(b, c).$$

Now we can associate, to this inner-product, the hermitian inner-product:

(3) $$\langle x, y\rangle=(x, \tilde{y}), \quad (x, y \in V)$$

and the corresponding norm $\|x\|=\langle x, x\rangle^{\frac{1}{2}}$, where $\tilde{y}$ denotes the complex-conjugation of y with respect to the real form V_R: $V_R=\{\alpha f_1+u^*+v+\beta f_2 \in V;\ \alpha, \beta \in R;\ u, v \in \mathfrak{J}_R$ (see [3])$\}$.

Let us now consider, among linear transformations of V, two special classes of them; namely $\mathfrak{D}=\mathfrak{D}(\mathfrak{V})$ denotes the derivation algebra of $\mathfrak{V}$, while $\mathfrak{L}=\mathfrak{L}(\mathfrak{D})$ the set of all left-translations $L(x)$ in V such that $\mathrm{Trace}\,(x)=0$. Then $\mathfrak{D}\cap\mathfrak{L}=\{0\}$, so we get the direct sum:

(4) $$\mathfrak{G}=\mathfrak{D}\oplus\mathfrak{L} \quad (\text{in } \mathfrak{gl}(V)).$$

$\mathfrak{G}$ is closed under the bracket operation in $\mathfrak{gl}(V)$. In fact we have

Proposition 1 (Brown [1]).

(i) $[D, L(x)]=L(Dx)$ for $D \in \mathfrak{D}$,

(ii) $[L(f_1-f_2), L(u)]=\dfrac{2}{3}L(u)$ *for* $u \in \mathfrak{J}$,

(iii) $[L(f_1-f_2), L(v^*)]=-\dfrac{2}{3}L(v^*)$ *for* $v^* \in \mathfrak{J}^*$,

(iv) $[L(u), L(v)]=[L(u^*), L(v^*)]=0$ *for* $u, v \in \mathfrak{J}$,

(v) $[L(u), L(v^*)]=(u, v)L(f_1-f_2)+E$;

where $E \in \mathfrak{D}$ *and is given by*

$$E=2\cdot R\left(-\frac{1}{3}(u, v)e+u\circ v\right)+2\cdot[R(u), R(v)]$$

(R *denotes the right translation in the algebra* $\mathfrak{J}$; *see* [2]).

Thus, $\mathfrak{G}$ is a complex linear Lie algebra which is turned out to be isomorphic to the complex simple Lie algebra of exceptional type E_7 [1], while $\mathfrak{D}$ is a subalgebra of $\mathfrak{G}$ and is isomorphic to the complex simple Lie algebra of type E_6. Furthermore the following holds

Proposition 2 (Brown [1]).

(i) $\mathfrak{D}(V_1)=0$, $\mathfrak{D}(V_4)=0$; $\mathfrak{D}(\mathfrak{J})\subset\mathfrak{J}$, $\mathfrak{D}(\mathfrak{J}^*)\subset\mathfrak{J}^*$;

(ii) *the representation of* $\mathfrak{D}$ *over* $\mathfrak{J}=V_3$ *is the irreducible one of* $\mathfrak{D}$ *in the sense of Chevalley and Schafer* [2], *and the representation of* $\mathfrak{D}$ *over* $\mathfrak{J}^*=V_2$ *is its contragredient one.*

4. In this section we describe a symmetric pair of $\mathfrak{G}$ corresponding to the irreducible bounded domain of type (VI); namely a symmetric pair of type EVII (see [6]):

Proposition 3. *A symmetric pair* $\mathfrak{G}=\mathfrak{K}\oplus\mathfrak{M}$ *of type* EVII *is given by*

$$\mathfrak{K}=\mathfrak{D}\oplus\{L(f_1-f_2)\},\quad \mathfrak{M}=\{L(u)+L(v^*)\,;\,u,v\in\mathfrak{J}\},$$

and a complex symmetric pair in the sense of [4] *is furnished with*

$$\mathfrak{M}=\mathfrak{N}^+\oplus\mathfrak{N}^-\,;\ \mathfrak{N}^+=\{L(u)\,;\,u\in\mathfrak{J}\},\ \mathfrak{N}^-=\{L(v^*)\,;\,v\in\mathfrak{J}\}\,;$$

namely $\mathfrak{N}^\pm$ *are naturally isomorphic to* $\mathfrak{J}$.

A compact form $\mathfrak{G}_u$ of $\mathfrak{G}$ will be given by the following

Proposition 4. $\mathfrak{G}_u$ *is the linear closure over* R *spanned by the following elements*:

$$\sqrt{-1}\,L(f_1-f_2),\ \sqrt{-1}\,L(u^*+u),\ L(u^*-u)\quad (u\in\mathfrak{J}_R),$$
$$\sqrt{-1}\,R(v)\ (v\in\mathfrak{J}_R),\ E\in\mathfrak{D}_R(\mathfrak{J}),$$

where the elements in the second line are generators of a compact form of $\mathfrak{D}$(*=the Lie algebra of type* E_6) (*see* [3]).

Hence, the complex-conjugation ι *of* $\mathfrak{G}$ *over* $\mathfrak{G}_u$ *can be, restricted on* $\mathfrak{M}=\mathfrak{N}^+\oplus\mathfrak{N}^-$, *expressed as below*:

$$\iota\,;\,L(u)\to -L(\tilde{u}^*),\ L(u^*)\to -L(\tilde{u})\ (u\in\mathfrak{J}).$$

Next, we denote by $\tilde{\rho}$ the representation of Brown described in § 3 and by ρ_K the restriction of $\tilde{\rho}$ to $\mathfrak{K}$, then (ρ_K, V) is completely reducible and is decomposed into irreducible components (ρ_i, V_i) $(1\leqslant i\leqslant 4)$ as in (1). In fact, both (ρ_1, V_1) and (ρ_4, V_4) are scaler representations which are explicitly observed from § 2, and (ρ_2, V_2) and (ρ_3, V_3) are described in Proposition 2 in § 3.

Furthermore we have to show the decomposition (1) of V satisfies the conditions claimed in [4], § 2.

Proposition 5. $\mathfrak{N}^+(V_1)=0,\ \mathfrak{N}^+(V_i)\subset V_{i-1}\ (2\leqslant i\leqslant 4)\,;\ \mathfrak{N}^-(V_i)\subset V_{i-1}$ $(1\leqslant i\leqslant 3),\ \mathfrak{N}^-(V_4)=0$.

Thus, as for the notation in [4], *we have* $p=n_1=1,\ r=n_2=27,$ $n_3=27,\ n_4=1,\ q=55$, *whence our domain* D *has to be realized in* $\mathfrak{J}\cong V_2^*$, *which is a complex vector space of dimension* 27.

5. Let $Z=L(u)\in\mathfrak{N}^+$ $(u\in\mathfrak{J})$. Then $Z^*=-\iota(Z)\in\mathfrak{N}^-$ is equal to $L(\tilde{u}^*)$ ($\tilde{u}$=the complex-conjugation of u with respect to $\mathfrak{J}_R$). According to the decomposition (1) of V, Z and Z^* are written in the following matrix-forms, taking suitable bases of V_i $(1\leqslant i\leqslant 4)$:

$$Z=\begin{pmatrix}0 & Z_1 & & \\ & & Z_2 & \\ & & & Z_3 \\ & & & 0\end{pmatrix},\quad Z^*=\begin{pmatrix}0 & & & \\ Z_1^* & & & \\ & Z_2^* & & \\ & & Z_3^* & 0\end{pmatrix}.$$

Hence, for $X_1\in\mathfrak{J}^*$, the adjoint operator $\theta[Z^*, Z]$ for $[Z^*, Z]\in\mathfrak{K}$ (see [4]) is

$$\theta[Z^*, Z]:\ X_1\to(Z_1Z_1^*+Z_1^*Z_1-Z_2Z_2^*)X_1,$$

where the linear mapping $Z_1:\mathfrak{J}^*\to C$ is identified with an element of $\mathfrak{J}$, $Z_1^*: C\to\mathfrak{J}^*$ with one of $\mathfrak{J}^*$, $Z_2:\mathfrak{J}\to\mathfrak{J}^*$ with one of $\mathfrak{J}^*\otimes\mathfrak{J}^*$, $Z_2^*:\mathfrak{J}\to\mathfrak{J}$

with one of $\mathfrak{J}\otimes\mathfrak{J}$, respectively. Therefore we may consider $Z_1Z_1^* \in C$, $Z_1^*Z_1 \in \mathfrak{gl}(\mathfrak{J}^*)$ and $Z_2Z_2^* \in \mathfrak{gl}(\mathfrak{J}^*)$; in fact, we have

$$Z_1Z_1^*: f_1 \to \|u\|^2 f_1 \quad (\|u\|^2=(u,\bar{u}))$$
$$Z_1^*Z_1: w^* \to (u, w)\bar{u}^*$$
$$Z_2Z_2^*: w^* \to 4(u\times(\bar{u}\times w))^* \quad (w^* \in \mathfrak{J}^*),$$

where the last two hermitian operators on $\mathfrak{J}^*$ can be identified with those on $\mathfrak{J}$ canonically; namely we may regard them as

$$Z_1^*Z_1: w \to (u, w)\bar{u}=(\bar{u}\otimes u^*)w$$
$$Z_2Z_2^*: w \to 4C_u\cdot C_{\bar{u}}(w) \quad (w \in \mathfrak{J}),$$

where C_u denotes the left translation in $\mathfrak{J}$ with respect to the crossed-product: $C_u(v)=u\times v$ for $v \in \mathfrak{J}$. Here we see easily that $C_{\bar{u}}=C_u^*$ (=the adjoint operator of C_u with respect to the hermitian inner-product (3) in $\mathfrak{J}$). Finally we conclude from Theorem 1 in [4] and the above that the canonical model of our symmetric domain D is given by $D=\{u \in \mathfrak{J}; \|u\|^2 I_{27}+(\bar{u}\otimes u^*)-4C_u\cdot C_u^* < 2I_{27}\}$, where we relpace u by $\sqrt{2}\cdot u$, and then we get the following result:

Theorem. *The irreducible bounded symmetric domain D of type* (VI) *is realized as*

$$D=\{u \in \mathfrak{J}; \|u\|^2 I+\bar{u}\otimes u^*-4C_u\cdot C_u^* < I\}.$$

We shall publish in a forthcoming paper the full proofs for all the statements in this note as well as those in the preceding note [3].

References

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