185. On the Univalence of Certain Integral

By Mamoru NUNOKAWA
Gunma University

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1. Let S be the class of functions $f(z)$ regular, univalent in $|z|<1$ and normalized by $f(0)=0$, $f'(0)=1$. On the other hand, let C, S^* and K be the subclass of S convex, starlike and close-to-convex functions, respectively. In the recent papers, [1], [2], [3, p. 40], [6, 7] and [9], the univalence of the functions

$$g(z)=\int_0^z \left(\frac{f(t)}{t}\right)^\alpha dt \qquad \text{and} \qquad g(z)=\int_0^z (f'(t))^\alpha dt$$

was studied.

2. On the univalence of $g(z)=\int_0^z (f'(t))^\alpha dt$.

Lemma 1. *Let $f(z)$ be regular for $|z|\leqq r$ and $f'(z)\neq 0$ on $|z|=r$. Suppose that on $|z|=r$*

$$\int_0^{2\pi} darg\ df(z)=\int_0^{2\pi}\frac{\partial}{\partial\theta}[\arg zf'(z)]d\theta=\int_0^{2\pi}\mathrm{Re}\left(1+\frac{zf''(z)}{f'(z)}\right)d\theta=2\pi$$

If furthermore

$$\int_{\theta_1}^{\theta_2} darg\ df(z)=\int_{\theta_1}^{\theta_2}\frac{\partial}{\partial\theta}[\arg zf'(z)]d\theta<3\pi \quad for \quad \theta_1<\theta_2$$

or

$$\int_{\theta_1}^{\theta_2} darg\ df(z)=\int_{\theta_1}^{\theta_2}\frac{\partial}{\partial\theta}[\arg zf'(z)]d\theta>-\pi \quad for \quad \theta_1<\theta_2$$

then $f(z)$ is univalent and close-to-convex in $|z|<r$.

We owe this lemma to Umezawa [11] and Reade [8].

Lemma 2. *Let $f(z)=z+\sum_{n=2}^{\infty}a_nz^n\in K$. Then there is a $\rho<1$ such that for all γ in the interval $\rho<\gamma<1$*

$$\int_{|z|=r} darg\ df(z)=2\pi$$

and

$$3\pi>\int_C darg\ df(z)>-\pi,$$

where C is an arbitrary arc on the boundary $|z|=r$.

We owe this lemma to Umezawa [12, Theorem 1].

Theorem 1. *Let $f(z)=z+\sum_{n=2}^{\infty}a_nz^n\in K$. Then*

$$g(z)=\int_0^z (f'(t))^\alpha dt$$

belongs to S and close-to-convex for $-\frac{1}{3} \leqq \alpha \leqq 1$.

Proof. From Lemma 2 there is a $\rho<1$ such that for all r in the interval $\rho<r<1$

(1) $$\int_{|z|=r} darg\ df(z)=2\pi$$

and

(2) $$3\pi>\int_C darg\ df(z)>-\pi,$$

where C is an arbitrary are on the boundary $|z|=r$.

It is easily obtained that

$$\int_C darg\ dg(z)=\alpha\int_C darg\ df(z)+(1-\alpha)\int_C darg\ dz$$

and therefore from (1)

$$\int_{|z|=r} darg\ dg(z)=\alpha\int_{|z|=r} darg\ df(z)+(1-\alpha)\int_{|z|=r} darg\ dz=2\pi.$$

Letting α be a negative real number, then we have from (2)

$$\int_C darg\ dg(z)\leqq\alpha\int_C darg\ df(z)+2\pi(1-\alpha)$$
$$<-\alpha\pi+2\pi(1-\alpha)=2\pi-3\alpha\pi.$$

Therefore we have

$$\int_C darg\ dg(z)<3\pi \qquad \text{for} \qquad -\frac{1}{3}\leqq\alpha<0.$$

Taking r sufficiently near to 1, it follows from Lemma 1 that $g(z)$ is univalent and close-to-convex in $|z|<1$ for $-\frac{1}{3}\leqq\alpha<0$.

Applying this result and [9, Theorem 1] we can complete our proof. This is an improvement of [9, Theorem 1].

Theorem 2. *Let* $f(z)=z+\sum_{n=2}^{\infty}a_nz^n\in C$. *Then*

$$g(z)=\int_0^z(f'(t))^\alpha dt$$

belongs to S and close-to-convex for $-0.5\leqq\alpha\leqq1.5$.

Proof. Since $f(z)$ is a convex function we have

$$\int_C darg\ df(z)>0$$

where C is an arbitrary arc on the boundary $|z|=r$ and so by the same method as in the proof of Theorem 1 we can prove from Lemma 1 that $g(z)$ is univalent and close-to-convex in $|z|<1$ for $-0.5\leqq\alpha<0$. Applying this result and [7, Theorem 3] we can complete our proof.

3. On the univalence of $g(z)=\int_0^z\left(\frac{f(t)}{t}\right)^\alpha dt$.

Lemma 3. *Let* $f(z)=z+\sum_{n=2}^{\infty}a_nz^n\in K$. *Then we have*

$$3\pi > \int_C darg\, f(z) > -\pi$$

where C is an arbitrary arc on the boundary $|z| = r < 1$.

Proof. Since $f(z)$ is close-to-convex, there exists by [4] a convex function $h(z)$ such that

$$\mathrm{Re}\frac{f'(z)}{h'(z)} > 0 \qquad \text{in} \quad |z| < 1.$$

Then we have

$$\mathrm{Re}\frac{f(z)}{h(z)} > 0 \qquad \text{in} \quad |z| < 1$$

by Sakaguchi [10], [5]. This shows that

$$\pi > \int_C darg\, \frac{f(z)}{h(z)} > -\pi$$

where C is an arbitrary arc on the boundary $|z| = r < 1$ and so that

$$3\pi \geqq \int_C darg\, h(z) + \pi > \int_C darg\, f(z) > \int_C darg\, h(z) - \pi \geqq -\pi.$$

This completes our proof.

Theorem 3. *Let* $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in K$. *Then*

$$g(z) = \int_0^z \left(\frac{f(t)}{t}\right)^{\alpha} dt$$

belongs to S and close-to-convex in $|z| < 1$ *for* $-\frac{1}{3} \leqq \alpha \leqq 1$.

Proof. Take r sufficiently near to 1. Then we have on $|z| = r < 1$

$$\int_{|z|=r} darg\, dg(z) = \int_{|z|=r} darg\, g'(z) + \int_{|z|=r} darg\, dz$$
$$= \alpha \int_{|z|=r} darg\, f(z) + (1-\alpha) \int_{|z|=r} darg\, dz = 2\pi.$$

Let α be a negative real number. Then we have from Lemma 3

$$\int_C darg\, dg(z) = \alpha \int_C darg\, f(z) + (1-\alpha) \int_C darg\, dz$$
$$< -\alpha\pi + 2\pi(1-\alpha).$$

Therefore we have

$$\int_C darg\, dg(z) < 3\pi \quad \text{for} \quad -\frac{1}{3} \leqq \alpha < 0.$$

This shows that $g(z)$ is univalent and close-to-convex in $|z| < 1$ for $-\frac{1}{3} \leqq \alpha < 0$. On the other hand, we have also

$$\int_C darg\, dg(z) > -\pi \quad \text{for} \quad 0 \leqq \alpha \leqq 1.$$

Therefore $g(z)$ is univalent and close-to-convex in $|z| < 1$ for $0 \leqq \alpha \leqq 1$. This completes our proof and improves [1, Theorem 1].

Theorem 4. *Let* $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in S^*$. *Then*

$$g(z)=\int_0^z\left(\frac{f(t)}{t}\right)^\alpha dt$$

belongs to S and close-to-convex in $|z|<1$ *for* $-0.5\leqq\alpha\leqq1.5$.

Proof. Take r sufficiently near to 1. Then we have on $|z|=r<1$

$$\int_{|z|=r} darg\ dg(z)=\alpha\int_{|z|=r} darg\ f(z)+(1-\alpha)\int_{|z|=r} darg\ dz=2\pi$$

Since $f(z)$ is starlike in $|z|<1$, we have

$$\int_C darg\ f(z)>0$$

where C is an arbitrary arc on the boundary $|z|=r$. Let α be a negative real number. Then we have

$$\int_C darg\ dg(z)=\alpha\int_C darg\ f(z)+(1-\alpha)\int_C darg\ dz$$
$$<(1-\alpha)\int_C darg\ dz\leqq2\pi(1-\alpha).$$

Therefore we have

$$\int_C darg\ dg(z)<3\pi \quad \text{for} \quad -0.5\leqq\alpha<0.$$

Applying Lemma 1 to $g(z)$ we have that $g(z)$ is univalent and close-to-convex in $|z|<1$ for $-0.5\leqq\alpha<0$. For the case in which α is a positive real number, we have also

$$\int_C darg\ dg(z)>-\pi \quad \text{for} \quad 0\leqq\alpha\leqq1.5$$

Therefore $g(z)$ belongs to S and close-to-convex in $|z|<1$ for $0\leqq\alpha\leqq1.5$. This completes our proof. By the same method as above we obtain the following theorems:

Theorem 5. *Let* $f(z)=z+\sum_{n=2}^{\infty}a_nz^n\in C$. *Then*

$$g(z)=\int_0^z\left(\frac{f(t)}{t}\right)^\alpha dt$$

is univalent and close-to-convex in $|z|<1$ *for* $-1\leqq\alpha\leqq3$.

Theorem 6. *Let* $f(z)=z+\sum_{n=2}^{\infty}a_nz^n\in S^*$. *Then*

$$g(z)=\int_0^z\left(\frac{tf'(t)}{f(t)}\right)^\alpha dt$$

is univalent and close-to-convex in $|z|<1$ *for* $-1\leqq\alpha\leqq1$.

Theorem 7. *Let* $f(z)=z+\sum_{n=2}^{\infty}a_nz^n$ *be regular, univalent and* $\mathrm{Re}\frac{f(z)}{z}>0$ *in* $|z|<1$ *and*

$$g(z)=\int_0^z\left(\frac{f(t)}{t}\right)^\alpha dt$$

Then $g(z)$ *is univalent and close-to-convex in* $|z|<1$ *for* $-1\leqq\alpha\leqq1$.

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