29. *Note on Covariance Operators of Probability Measures on a Hilbert Space*

By D. KANNAN and A. T. BHARUCHA-REID*)
Center for Research in Probability
Wayne State University, Detroit, Michigan

(Comm. by Kinjirô KUNUGI, M. J. A., Feb. 12, 1970)

1. Introduction. Let $(\Omega, \mathcal{A}, \mu)$ be a probability measure space, and let $(\mathfrak{H}, \mathcal{B})$ denote a measurable space where $\mathfrak{H}$ is a real separable Hilbert space with inner product $\langle\cdot, \cdot\rangle$, and $\mathcal{B}$ is the σ-algebra of Borel subsets of $\mathfrak{H}$. Let $x(\omega)$ denote a $\mathfrak{H}$-valued random variable, that is $\{\omega : x(\omega) \in B\} \in \mathcal{A}$ for all $B \in \mathcal{B}$; and let ν_x denote the probability measure (or distribution) on $\mathfrak{H}$ induced by μ and x, that is $\nu_x = \mu \circ x^{-1}$, or $\nu_x(B) = \mu(x^{-1}(B))$ for all $B \in \mathcal{B}$. Let $\mathfrak{M}(\mathfrak{H})$ denote the space of all probability measures on $\mathfrak{H}$; and let $\nu \in \mathfrak{M}(\mathfrak{H})$ be such that $\varepsilon_\nu\{\|x\|^2\} = \int \|x\|^2 d\nu < \infty$. Then the *covariance operator* S of ν is defined by the equation

$$\langle Sg, g\rangle = \int_{\mathfrak{H}} \langle f, g\rangle^2 d\nu(f) \tag{1}$$

(cf. Grenander [1], Parthasarathy [4], Prokhorov [5]). A linear operator L in $\mathfrak{H}$ is said to be an *S-operator* if it is a positive, self-adjoint operator with finite-trace; hence L is compact. S-operators play a fundamental role in the study of probability theory in Hilbert spaces (cf. [2, 3, 6, 10]). We recall that the function

$$\hat{\nu}(g) = \exp\{-1/2 \langle Sg, g\rangle\}, \quad g \in \mathfrak{H}, \tag{2}$$

is the *characteristic functional* (or Fourier transform) of a probability measure on $\mathfrak{H}$ iff S is an S-operator. Also, if ν is the measure corresponding to $\hat{\nu}$, then $\varepsilon_\nu\{\|x\|^2\} < \infty$; and S is the covariance operator of ν. We also recall that a measure ν on $\mathfrak{H}$ is *normal* (or *Gaussian*) iff $\hat{\nu}$ is of the form

$$\hat{\nu}(g) = \exp\{i\langle g_0, g\rangle - 1/2\langle Sg, g\rangle\}, \tag{3}$$

where g_0 is a fixed element in $\mathfrak{H}$ and S is an S-operator. The element g_0 is the expectation of ν, and S its covariance operator.

Let $L_2(\Omega, \mathcal{A}, \mu, \mathfrak{H}) = L_2(\Omega, \mathfrak{H})$ denote the space of $\mathfrak{H}$-valued random variables $x(\omega)$ such that $\varepsilon_\mu\{\|x\|^2\} < \infty$, with norm defined by

$$[x]_2 = (\varepsilon_\mu\{\|x\|^2\})^{1/2}. \tag{4}$$

For any finite sequences $\{\xi_i\} \subset L_2(\Omega, \mathcal{A}, \mu) = L_2(\Omega)$ and $\{f_i\} \subset \mathfrak{H}$, put

$$\sum_{i=1}^n \xi_i(\omega) \odot f_i = \sum_{i=1}^n \xi_i(\omega) f_i (\text{mod } \mu). \tag{5}$$

*) Research supported by National Science Foundation Grant No. GP-13741.

The above relation defines an element of $L_2(\Omega, \mathfrak{H})$. Let $L_2(\Omega)\odot\mathfrak{H}$ denote the algebraic tensor product of the Hilbert spaces $L_2(\Omega)$ and $\mathfrak{H}$; that is $L_2(\Omega)\odot\mathfrak{H}$ is the set of all functions defined by (5); and it is also a dense linear subspace of $L_2(\Omega, \mathfrak{H})$ with norm $[\cdot]_2$. This norm is a *crossnorm* (cf. Schattan [7], p. 28), that is, $[\xi\odot f]_2 = \|\xi\|_2 \cdot \|f\|$, $\xi \in L_2(\Omega)$, $f \in \mathfrak{H}$. Let $L_2(\Omega)\hat{\otimes}\mathfrak{H}$ denote the tensor product Hilbert space which is the completion of $L_2(\Omega)\odot\mathfrak{H}$ with respect to the norm defined by (4); that is $L_2(\Omega, \mathfrak{H}) = L_2(\Omega)\hat{\otimes}\mathfrak{H}$. Since $\nu_x = \mu\circ x^{-1}$, it is clear that those elements $x \in L_2(\Omega)\hat{\otimes}\mathfrak{H}$ generate measures $\nu_x \in \mathfrak{M}(\mathfrak{H})$ for which covariance operators are defined. In the present note we use two theorems of Umegaki and Bharucha-Reid ([9], Sections 4 and 5) on a class of operators associated with elements of a tensor product Hilbert space to obtain representations of covariance operators.

2. Representations of covariance operators. Let H and $\mathfrak{H}$ be two real separable Hilbert spaces with inner products $(\cdot, \cdot)$ and $\langle \cdot, \cdot \rangle$, respectively; and let $H\odot\mathfrak{H}$ denote the algebraic tensor product of H and $\mathfrak{H}$. For any two elements $x = \sum_{i=1}^{n} \xi_i \odot f_i$ and $y = \sum_{j=1}^{m} \eta_j \odot g_j$, where $\xi_i, \eta_j \in H$ and $f_i, g_i \in \mathfrak{H}$, put

$$\langle x | y \rangle = \sum_{i=1}^{n} \sum_{j=1}^{m} (\xi_i, \eta_j)\langle f_i, g_j \rangle. \tag{6}$$

Then, $\langle \cdot | \cdot \rangle$ is an inner product in $H\odot\mathfrak{H}$; and

$$[x]_2 = \left\langle \sum_{i=1}^{n} \xi_i \odot f_i \,\middle|\, \sum_{i=1}^{n} \xi_i \odot f_i \right\rangle^{1/2} \tag{7}$$

satisfies the norm condition on $H\odot\mathfrak{H}$. Let $H\hat{\otimes}\mathfrak{H}$ denote the completion of $H\odot\mathfrak{H}$ with respect to the norm defined by (7); then $H\hat{\otimes}\mathfrak{H}$ is the tensor product Hilbert space of H and $\mathfrak{H}$. For $x, y \in H\odot\mathfrak{H}$ (where x and y are as defined as above) and $\psi_1, \psi_2 \in \mathfrak{H}$, put

$$F_{x,y}(\psi_1, \psi_2) = \sum_{i,j=1}^{n,m} (\xi_i, \eta_j)\langle f_i, \psi_2 \rangle \langle \psi_1, g_j \rangle. \tag{8}$$

Then $F_{x,y}$ is a bounded bilinear functional on $\mathfrak{H}$; and there exists a unique bounded operator, say $S_{x,y}$, in $\mathfrak{H}$ such that $\langle S_{x,y}\psi_1, \psi_2 \rangle = F_{x,y}(\psi_1, \psi_2)$. The operator $S_{x,y}$ has been defined for every pair $x, y \in H\odot\mathfrak{H}$; but $S_{x,y}$ is defined also for any pair $x, y \in H\hat{\otimes}\mathfrak{H}$; since for $x, y \in \hat{\otimes}\mathfrak{H}$ there exists sequences $\{x_n\}, \{y_n\} \subset H\odot\mathfrak{H}$ such that $\|x_n - x\| \to 0$, $\|y_n - y\| \to 0$, and S_{x_n, y_n} converges in trace norm to a trace class operator $S_{x,y}$ which is independent of the choice of the sequences $\{x_n\}$ and $\{y_n\}$.

We now state the following result:

Theorem A (Theorem 4.1 of [9]). *For every pair $x, y \in H\hat{\otimes}\mathfrak{H}$, there exists a unique trace class operator $S_{x,y}$ which is conjugate bilinear in x, y satisfying* (i) $S_x = S_{x,x} \geq 0$, (ii) $S^*_{x,y} = S_{y,x}$, (iii) $\mathrm{Tr}[S_{x,y}] = \langle x | y \rangle$, (iv) *Uniform norm* $\|S_{x,y}\| \leq$ *Trace norm* $[S_{x,y}] \leq \|x\| \cdot \|y\|$, *and* (v) $S_{x,y}$ *is completely positive*; *that is, for any finite sequences*

$\{x_i\}\subset H\hat{\otimes}\mathfrak{H}$ *and* $\{z_i\}\subset\mathfrak{H}$, $\sum_{i,j}\langle z_i, S_{x_i,x_j}z_j\rangle\geq 0$.

In this note we are concerned only with the case $S_x=S_{x,x}$; hence in this case S_x is a positive, self-adjoint operator with finite trace, and S_x is an S-operator.

Let $\mathfrak{H}$ be a real separable Hilbert space, and let $H=L_2(\Omega, \mathcal{A}, \mu)$. In this case $L_2(\Omega, \mathfrak{H})=L_2(\Omega)\hat{\otimes}\mathfrak{H}$. We now prove the following representation theorem.

Theorem. *For every* $x\in L_2(\Omega,\mathfrak{H})$ *there is probability measure* ν_x *on* $\mathfrak{H}$ *such that the S-operator* S_x *is the covariance operator of* ν_x; *that is* S_x *admits the representation*

$$\langle S_x g, g\rangle=\int_{\mathfrak{H}}\langle f, g\rangle^2 d\nu_x(f). \tag{9}$$

Proof. Let $x\in L_2(\Omega)\odot\mathfrak{H}$, i.e. $x=\sum_{i=1}^{k}\xi_i\odot h_i$, where $\xi_i\in L_2(\Omega)$ is a real-valued random variable, and $h_i\in\mathfrak{H}$. Now

$$\begin{aligned}\langle S_x g, g\rangle&=\sum_{i,j=1}^{k}(\xi_i,\xi_j)\langle h_i, g\rangle\langle g, h_j\rangle\\ &=\sum_{i,j=1}^{k}\left[\int_\Omega \xi_i(\omega)\xi_j(\omega)d\mu(\omega)\right]\langle h_i, g\rangle\langle g, h_j\rangle\\ &=\int_\Omega\sum_{i,j=1}^{k}\langle h_i, g\rangle\langle g, h_j\rangle\xi_i(\omega)\xi_j(\omega)d\mu(\omega)\\ &=\int_\Omega\sum_{i,j=1}^{k}\langle \xi_i(\omega)h_i, g\rangle\langle g, \xi_j(\omega)h_j\rangle d\mu(\omega)\\ &=\int_\Omega\left\langle\sum_{i=1}^{k}\xi_i(\omega)\odot h_i, g\right\rangle^2 d\mu(\omega)\\ &=\int_\Omega\langle x(\omega), g\rangle^2 d\mu.\end{aligned}$$

Hence

$$\langle S_x g, g\rangle=\int_\Omega\langle x(\omega), g\rangle^2 d\mu(\omega) \tag{10}$$

From the definition of the probability measure ν_x, for every measurable function φ on $\mathfrak{H}$,

$$\int_{\mathfrak{H}}\varphi d\nu_x=\int_\Omega(\varphi\circ x)d\mu. \tag{11}$$

we can take φ as the continuous function on $\mathfrak{H}$ given by $\varphi(f)=\langle f, g\rangle^2$, for fixed g. Hence

$$\int_{\mathfrak{H}}\langle f, g\rangle^2 d\nu_x(f)=\int_\Omega\langle x(\omega), g\rangle^2 d\mu(\omega). \tag{12}$$

Using (12) in (10) we obtain (9) for any $x\in L_2(\Omega)\odot\mathfrak{H}$.

Now let $x\in L_2(\Omega,\mathfrak{H})$. Then there exists a sequence $x_n\in L_2(\Omega)\odot\mathfrak{H}$ such that $[x_n-x]_2\to 0$, where $[\cdot]_2$ is the crossnorm defined by (4). This implies that $S_{x_n}\to S_x$ in trace norm, and $\langle S_{x_n}g, g\rangle\to\langle S_x g, g\rangle$. We also have $\int_\Omega\langle x_n(\omega), g\rangle^2 d\mu\to\int_\Omega\langle x(\omega), g\rangle^2 d\mu$. Hence from (10) and (12) we obtain (9) for any $x\in L_2(\Omega,\mathfrak{H})$.

We now consider another approach to the representation of S-operators. Let x and y be two given elements of $\mathfrak{H}$. The tensor product $x\otimes y$ represents an operator on $\mathfrak{H}$ whose defining equation is given by $(x\otimes y)z=\langle z, y\rangle x$ for every $z\in\mathfrak{H}$ (cf. Schattan [7], p. 69; [8], p. 7). The following result is utilized:

Theorem B (Theorem 5.1 of [9]). *For any pairs $x, y\in L_2(\Omega,\mathfrak{H})$ and $f, g\in\mathfrak{H}$*

$$\langle S_{x,y}f, g\rangle=\int_\Omega \mathrm{Tr}[x(\omega)\otimes y(\omega)\cdot f\otimes g]d\mu(\omega), \tag{13}$$

and the integrand in (13) *is measurable.*

As before, we restrict our attention to the case $S_x=S_{xx}$, and take $\mathfrak{H}$ to be a real separable Hilbert space. Using the fact that $(f_1\otimes f_2)(g_1\otimes g_2)=\langle g_1, f_2\rangle f_1\otimes g_2$, we have $\mathrm{Tr}[x(\omega)\otimes x(\omega)\cdot g\otimes g]=\langle x(\omega), g\rangle^2$. Hence (13) becomes $\langle S_x g, g\rangle=\int_\Omega\langle x(\omega), g\rangle^2 d\mu(\omega)$, which is (10). Utilizing (11) and (12), we obtain (9) for all $x\in L_2(\Omega,\mathfrak{H})$.

3. Examples and applications. In this section we mention a few applications of the above representations and compute the covariance operators associated with certain random functions.

a. An obvious application is to the characteristic functionals of probability measures in $\mathfrak{M}(\mathfrak{H})$; for example, it follows from (2) that $\hat{\nu}_x(g)$, the characteristic functional of a probability measure $\nu\in\mathfrak{M}(\mathfrak{H})$ induced by $x(\omega)$, is of the form

$$\hat{\nu}_x(g)=\exp\left\{-\frac{1}{2}\sum(\xi_i,\xi_j)\langle h_i, g\rangle\langle g, h_j\rangle\right\},\ g\in\mathfrak{H} \tag{14}$$

where $x(\omega)=\sum_{i=1}^{k}\xi_i(\omega)\odot h_i$, with $\xi_i(\omega)\in L_2(\Omega)$, $h_i\in\mathfrak{H}$.

b. In the study of random equations in Hilbert spaces we frequently encounter transformations of the form $y(\omega)=L[x(\omega)]$, where $x(\omega)$ is a Gaussian random variable and L is an endomorphism of $\mathfrak{H}$. If m_x and S_x denote the expectation of x and the covariance operator of the measure induced by ν_x respectively; then it is well-known (cf. [1], pp. 141–142) that $m_y=Lm_x$ and $S_y=LS_xL^*$. Hence, given the representation of S_x, an explicit representation of S_y can be obtained.

c. Let $\mathfrak{H}=L_2(T,\Theta,\tau)$ where $T=[0,1]$, Θ is the σ-algebra of Borel subsets of T, and τ is Lebesque measure on Θ. Let $L_2(\Omega,\mathfrak{H})$ denote the space of all $\mathfrak{H}$-valued measurable random functions $x=\{x(t,\omega), t\in T\}$ such that $\int_\Omega\|x\|^2d\mu<\infty$. In this case the tensor product Hilbert space is $L_2(\Omega,\mathfrak{H})=L_2(\Omega)\hat{\otimes}\mathfrak{H}(T)=L_2(\Omega\times T)$. Since x is a second-order random function its covariance kernel is of the form

$$R_x(s,t)=\int_\Omega x(s,\omega)x(t,\omega)d\mu(\omega). \tag{15}$$

An easy consequence of the representation (13) is that the covariance operator S_x on $L_2(T)$ is of the form

$$(S_x f)(s)=\int_T R_x(s,t)f(t)d\tau(t), \quad f\in L_2(T). \tag{16}$$

Also, we have $\mathrm{Tr}[S_x]=([x]_2)^2=\int_\Omega (\|x\|_2)^2 d\mu(\omega)=\int_T R_x(s,s)d\tau(s)=\mathrm{Tr}[R_x]$.

We now assume that x is continuous in quadratic mean. In this case the covariance kernel $R_x(s,t)$ is of the form

$$R_x(s,t)=\sum_{i=1}^{\infty}\frac{\varphi_i(s)\varphi_i(t)}{\lambda_i} \tag{17}$$

where the λ_i are the eigenvalues and the φ_i are the eigenfunctions of $R_x(s,t)$. Inserting (17) and (16) we have

$$\begin{aligned}(S_x f)(s)&=\int_T\sum_{i=1}^{\infty}\frac{\varphi_i(s)\varphi_i(t)}{\lambda_i}f(t)d\tau(t)\\ &=\sum_{i=1}^{\infty}\frac{\varphi_i(s)}{\lambda_i}\int_T\varphi_i(t)f(t)d\tau(t)=\sum_{i=1}^{\infty}\alpha_i\varphi_i(s),\end{aligned}$$

where $\alpha_i=\lambda_i^{-1}\langle\varphi_i,f\rangle$. This also follows from the representation of operator S_x; namely $S_x=\sum_{i=1}^{\infty}\lambda_i^{-1}\varphi_i\otimes\varphi_i$.

d. Let $\mathfrak{H}=l_2$; and let $x\in L_2(\Omega)\hat{\otimes}l_2$. In this case x is an l_2-valued random variable; and can be considered as a sequence $\{x_n(\omega)\}$ of real-valued random variables such that $\sum_{n=1}^{\infty}|x_n|^2<\infty$. Define $R_{ij}^{(x)}=\varepsilon\{x_i x_j\}$. Then, for $g=\{g_n\}\in l_2$,

$$\begin{aligned}\langle S_x g,g\rangle&=\int_\Omega\langle x(\omega),g\rangle^2 d\mu(\omega)=\int_\Omega\sum_{i,j=1}^{\infty}x_i(\omega)x_j(\omega)g_i g_j d\mu(\omega)\\ &=\sum_{i,j=1}^{\infty}g_i R_{ij}^{(x)}g_j.\end{aligned}$$

References

[1] U. Grenander: Probabilities on Algebraic Structures. John Wiley and Sons, New York (1963).

[2] R. Jajte: On the probability measures in Hilbert spaces. Studia Math., **29**, 221–241 (1968).

[3] N. P. Kandelaki and V. V. Sasonov: On a central limit theorem for random elements with values in Hilbert space (in Russian). Teor. Verojatnost. i Primenen., **9**, 43–52 (1964).

[4] K. R. Parthasarathy: Probability Measures on Metric Spaces. Academic Press, New York (1967).

[5] Yu. V. Prokhorov: Convergence of random processes and limit theorems in probability theory (in Russian). Teor. Verojatnost. i. Primenen., **1**, 177–238 (1956).

[6] K. Reinschke: Zum zentralen Grenzwertsatz für zufällige Elemente mit Werten aus einem Hilbertraum. Z. Wahrscheinlichkeitstheorie und Verw. Gebiete., **6**, 161–169 (1966).

[7] R. Schattan: A Theory of Cross-Spaces. Princeton University Press, Princeton, N. J. (1950).
[8] ——: Norm Ideals of Completely Continuous Operators. Springer-Verlag, Berlin (1960).
[9] H. Umegaki and A. T. Bharucha-Reid: Banach space-valued random variables and tensor products of Banach spaces. J. Math. Anal. Appl. (in press).
[10] S. R. S. Varadhan: Limit theorems for sums of independent random variables with values in a Hilbert space. Sankhya, **24**, 213–238 (1962).