25. *Sequential Convergence of Operators on Orlicz Spaces of Lebesgue-Bochner Measurable Functions in Various Operator Topologies and Some Applications*[*)]

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In an earlier paper [4], a complete set of representatives for the bounded, linear operators from an Orlicz space of Lebesgue-Bochner measurable functions to any Banach space was established. Presently, we are concerned with applying some of those results in order to establish criteria for the convergence of sequences of operators in various, frequently used operator topologies. Some of these results are extensions of some results in [1]; others are seemingly new even for the case considered therein.

We keep the same notation as in [4], and assume throughout that the generating convex function p satisfies the growth condition $p(2u) \leqslant cp(u)$ for some $c \geqslant 0$ and all $u \geqslant 0$.

Our first result entends those of [1] involving the convergence of a sequence of operators on the Orlicz space $L_p(v, Y)$ to the Banach space Z relative to the strong operator topology on $B(L_p(v, Y); Z)$. Recall that this topology is that of simple convergence in the terminology of [9]. Throughout we suppose that $T_n \in B(L_p(v, Y); Z)$ and $\mu_n \in M_q(V, B(Y; Z))$ are such that for $n \geqslant 0$:

$$T_n(f) = \int f d\mu_n$$

for all $f \in L_p(v, Y)$.

Theorem 1. $T_n \to T_0$ *(strongly) if and only if* $\{\mu_n\}$ *is bounded in* $M_q(V, B(Y; Z))$ *and for each* $A \in V$, $\mu_n(A) \to \mu_0(A)$ *(strongly) (relative to* $B(Y; Z)$*).*

Proof. Necessity follows from the inequality $\|\mu_n\|_{q,v} \leqslant 2\|T_n\|$ and the Banach-Steinhaus theorem.

To prove sufficiency, we note that the boundedness of $\{\mu_n\}$ and the inequality $\|T_n\| \leqslant \|\mu_n\|_{q,v}$ yields the boundedness of $\{\|T_n\|\}$. Thus it is enough that we show that $T_n g \to T_0 g$ for g belonging to some total family in $L_p(v, Y)$. By Theorem 8 of [3], the family $\{\chi_A y : A \in V, y \in Y\}$ is total in $L_p(v, Y)$; a simple calculation combined with the condition

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$\mu_n(A) \to \mu_0(A)$ (strongly) now completes the proof of sufficiency.

Recall from [4], that $M_q(V, W)$ may be defined for any Banach space W; simply identify W with $B(K; W)$, where K is the scalar field (=real of complex field) underlying W. With this identification in mind, using the representation theorem of $M_q(V, W)$ as $B(L_p(v, K); W)$ we readily see that $M_q(V, W)$ is a Banach space whose norm is given by

$$\|\mu\|_{q,v} = \sup\{|\Sigma_1^n a_i \mu(A_i)|\}$$

where $n \in N$, $a_1, \cdots, a_n \in K$, $A_1, \cdots, A_n \in V$ and are disjoint and $p(|a_1|)v(A_1) + \cdots + p(|a_n|)v(A_n) \leqslant 1$. In particular, in case W is itself the space K, we have that the norm in $M_q(V, K)$ can be given by $\|\mu\|_{q,v} = \sup\{\Sigma_1^n a_i |\mu(A_i)|\}$ where $a_1, \cdots, a_n$ can now be supposed $\geqslant 0$ and the rest supposed to be as before. (The author wishes to thank Professor W. M. Bogdanowicz for this observation, made by him in a seminar of which he was moderator.)

In light of the above remarks the following type of dominated convergence theorem is obtained as an easy application of Theorem 1.

Theorem 2. *Suppose that* $\mu_n \in M_q(V, B(Y : Z))$ *for* $n = 1, 2, \cdots$. *Suppose further that for each* $A \in V$, $\{\mu_n(A)\}$ *is a strongly convergent sequence in* $B(Y; Z)$, *with strong limit* $\mu_0(A)$. *Let* $\mu \in M_q(V, R)$ *be such that*

$$|\mu_n(A)| \leqslant \mu(A)$$

for all $n \geqslant 1$ *and all* $A \in V$.

Then $\mu_0 \in M_q(V; B(Y; Z))$ *and* $\int f d\mu_n \to \int f d\mu_0$ *for all* $f \in L_p(v, Y)$.

We now consider the problem of uniform convergence for sequences of operators. Here of course we know Theorem 2 of [4] yields that $T_n \to T_0$ (uniformly) if and only if $\mu_n \to \mu_0$ in $\|\ \|_{q,v}$. Thus, the question of characterizing uniform convergence is the same as that of characterizing norm convergence in $M_q(V, B(Y; Z))$. This is done by the next result (Theorem 3) which bears surprising similarily to Theorem 2 of (3):

Theorem 3. *Suppose for* $n \geqslant 1$ *that* $\mu_n \in M_q(V; B(Y; Z))$. *Then the following are equivalent*:

(1) $\mu_0 \in M_q(V, B(Y; Z))$ *and* $\|\mu_n - \mu_0\|_{q,v} \to 0$;

(2) $\{\mu_n\}$ *is* $\|\ \|_{q,v}$-*Cauchy and given* $A \in V$, $\mu_n(A) \to \mu_0(A)$ (*uniformly in* $B(Y; Z)$)

(3) $\{\mu_n\}$ *is* $\|\ \|_{q,v}$-*Cauchy and given* $A \in V$, *there exists a subsequence* $\{\mu_{n_k}\}$ *of* $\{\mu_n\}$ *such that* $\mu_{n_k}(A) \to \mu_0(A)$ (*uniformly*) *in* $B(Y; Z)$.

Proof. (1) implies (2) is an immediate consequence of the inequality:

$$|(\mu_n - \mu_0)(A)| \leqslant \|\chi_A\|_{p,v} \|\mu_n - \mu_0\|_{q,v}$$

That (2) implies (3) is patent. To see that (3) implies (1), suppose $\{\mu_n\}$ is $\|\ \|_{q,v}$-Cauchy, and let $\mu_0' \in M_q(V; B(Y; Z))$ be such that $\|\mu_n - \mu_0'\|_{q,v}$

$\to 0$: By (1) implies (2), for each A V,

$$\mu_n(A) \to \mu_0'(A)$$

in the uniform (norm) topology of $B(Y;Z)$. But if $A \in V$ then there is, by (3), subsequence $\{\mu_{n_k}\}$ of $\{\mu_n\}$ converging uniformly to $\mu_0(A)$. Thus, $\mu_0'(A) = \mu_0(A)$ and $\mu_0 \in M_q(V, B(Y;Z))$ with $\|\mu_n - \mu_0\|_{q,v} \to 0$.

We are now able to prove the following:

Theorem 4. *Suppose (p_1, q_1) and (p_2, q_2) are pairs of complementary Young's functions with p_1 and p_2 both satisfying the growth condition Δ_2. Suppose Z_1 and Z_2 are Banach spaces and that $T \in B(Z_1; Z_2)$ is such that the operator $\hat{T}$ defined by $(\hat{T}\mu)(A) = T(\mu(A))$ is well-defined from $M_{q_1}(V, B(Y;Z_1))$ to $M_{q_2}(V, B(Y;Z_2))$ where $(\hat{T}\mu)(A)(y) = T(\mu(A))y$. Then T is continuous.*

Proof. It suffices to apply the Closed Graph theorem and Theorem 3.

The following consequence of Theorem 4 is included if for no other reason than it's true and perhaps of some independent interest.

Theorem 5. *Suppose $Z_1, \cdots, Z_n$ are Banach spaces and $(p_1, q_1), \cdots, \cdots, (p_n, q_n)$ are complementary Young's functions (p_i as usual). Let $T: Z_1 \times \cdots \times Z_n \to Z$ be n-linear and continuous. Let $(X_1, V_1, v_1), \cdots, \cdots, (X_n, V_n, v_n)$ be volume spaces with (X, V, v) their product volume space (see [2]). Then, if T is well defined on $\prod_{i=1}^{n} M_{q_i}(V_i, B_i)$ where $B_i = B(Y;Z_i)$ to $M_q(V, B(Y;Z))$ by*

$$T(\mu_1, \cdots, \mu_n)(A)y = T(\mu_1(A_1)y, \cdots, \mu_n(A_n)y)$$

for $A = A_1 \times \cdots \times A_n \in V$, $y \in Y$, then T is likewise n-linear, continuous.

Some remarks are probably worth making. For one, we mention that while it is "natural" to expect or hope that the hypotheses of Theorems 4 and 5 always hold, it is not at all clear that this is true. In another paper—of considerably more general nature—the author will deal in some detail with the question of which operators (both linear and multilinear) "lift" in the manner described above. It seems that the question is intimately connected with summability theory. As an example, the following can be shown rather readily: If $T \in B(Y;Z)$ is a nuclear operator (see [6]), then $\hat{T}$ is well-defined between any pair of spaces.

Another remark is appropriate: many of the constructions basic to the establishment of the results of the present paper are valid in more general classes of locally convex linear topological spaces and therein a much richer theory both from the view of theory and application is possible.

Finally, of particular interest is the question of under what conditions on $T \in B(Y;Z)$ does the "lifting" described in Theorem 4 preserve the properties of Radon-Nikodym differentiability of various vector-valued measures. (Note: under fairly minimal conditions on the complementary pair (p, q) the reasonings of [1] can be mimicked yielding countable additivity of members of $M_q(V, Z)$; the conditions needed are clear upon a close look at [1] and [10]). The paper [8] of Rieffel seems to indicate an intimate connection between this property and that of the compactness T.

Lastly, of some interest insofar as sequential convergence of operators is concerned is that of weak-convergence. In this connection recall that a sequence of operators $C_n \in B(Y; Z)$ converges weakly to $C_0 \in B(Y; Z)$ when for each $z' \in Z'$ and each $y \in Y$, $(z' \circ C_n)(y)$ converges to $(z' \circ C_0)(y)$. It is readily seen that for bounded sequences $\{T_n\}$ from $B(L_p(v, Y); Z)$ that the weak convergence of T_n to T_0 is entirely equivalent to that of $\mu_n(A)$ to $\mu_0(A)$ for each $A \in V$. All that one needs is to note that if $z' \in Z'$ then with the obvious definition of $z' \circ \mu$, $z' \circ \mu \in M_q(V, K)$ whenever μ belongs to $M_q(V, B(Y; Z))$ and

$$z'\left(\int f\, d\mu\right) = \int f d(z' \circ \mu)$$

a relation easily cheked by looking at simple functions $f \in L_p(v, Y)$ and applying Theorem 8 of [3].

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