23. *Every C-Symmetric Banach $*$-Algebra is Symmetric*[1)]

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Since Gelfand and Naimark [1] had conjectured the symmetry of B^*-algebras, I. Kaplansky raised the more general question: is a C-symmetric Banach $*$-algebra symmetric?, and it remained open during the past twenty years. One knew that an intrinsic key to this problem is to prove that the sum of positive elements is also positive (for B^*-algebras, see [2], [5]). Recently we have proved in [9] that in a Banach $*$-algebra with the norm condition $\alpha\|x^*\|\|x\|\leq\|x^*x\|$ $(\alpha>0)$, the positive elements form a positive cone, and then it turned out that the method employed there can be applied for C-symmetric Banach $*$-algebras by a slight modification. Meantime, we have heard that S. Shirali and J. Ford [8] have solved Kaplansky's problem in the affirmative, that is, the following result has been established.

Theorem. *A C-symmetric Banach $*$-algebra is necessarily symmetric.*

In this paper we will supply a technically simple and possibly quick proof of the theorem by an adequate improvement of our previous work [9].

1. Let us recall that a Banach $*$-algebra A is symmetric if x^*x is quasi-regular for all x in A; it is C-symmetric if every closed commutative $*$-subalgebra is symmetric. In case A has a unit e, the symmetry means that $e+x^*x$ is invertible for every x in A and therefore the C-symmetry means that $e+x^*x$ is invertible for every normal element x (i.e., $x^*x=xx^*$) in A. Throughout this paper we shall mainly concern a (complex) Banach $*$-algebra A with unit e. We denote by $\sigma(x)$ the spectrum of an element x in A; a self-adjoint element h in A is said to be positive (strictly positive) if $\sigma(x)\subset[0,\infty)$ $((0,\infty))$, respectively, and then for self-adjoint elements h, k in A, we understand the symbol $h\leq k$ (or $h<k$) as usual. For a normal element x in A, $A(x)$ always means a maximal commutative $*$-subalgebra of A containing x, and we should recall that $A(x)$ is automatically closed. In the proof of the theorem, the fact that a strictly positive element in A has a (strictly) positive square root will play a relevant role (cf. [3], [10]).

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Lemma 1. *Let h be a strictly positive element in a Banach $*$-algebra A with unit. Then there exists a positive element k in $A(h)$ such that $h=k^2$.*

2. Let us consider a symmetric commutative Banach $*$-algebra A with unit e. Then the Gelfand representation of A may be stated as follows: A is $*$-isomorphic with a dense $*$-subalgebra of the algebra $C(\Omega)$ of all continuous functions on Ω, where Ω is the spectrum of A, that is, it is the compact Hausdorff space consisting of all multiplicative linear functionals φ on A with $\varphi(e)=1$. Moreover, let us recall that for an element x in A and a scalar λ, there is $\varphi \in \Omega$ such that $\varphi(x)=\lambda$ if and only if $\lambda \in \sigma(x)$. As an immediate consequence of the above statements, the C-symmetry of a Banach $*$-algebra A with unit is indeed equivalent to the fact that every self-adjoint element h in A has a real spectrum.

It is well known that if a C-symmetric Banach $*$-algebra A has no unit, then the Banach $*$-algebra A' obtained by adjunction of a unit to A is also C-symmetric, and further A is symmetric if and only if A' is symmetric (see [7]). Therefore, to prove the theorem, we may discuss only a C-symmetric Banach $*$-algebra with unit. In what follows, A will always mean a C-symmetric Banach $*$-algebra with unit e. Since $A(x)$ is symmetric for a normal element x in A, the Gelfand representation of $A(x)$ mentioned above can be extensively used. The following lemma is elementary, but substantially important in our treatment.

Lemma 2. *If $x^2=0$ in A, then $x^*x\geq 0$.*

In fact, the equality $e+(x+x^*)^2=(e+x^*x)(e+xx^*)$ yields that $e+x^*x$ is invertible. It follows from this that $e+\lambda x^*x$ is invertible for every $\lambda>0$.

Lemma 3. *The sum of positive elements in A is also positive.*

Proof. Following the procedure as in [6; IX, p. 302], our assertion can be reduced to show that for any invertible element x in A, x^*x is positive.[2)]

Now consider the algebra $A_0=A(x^*x)$ and the Gelfand representation $a\to\hat{a}$ of A_0 to $C(\Omega)$ as stated in the preceding paragraph. Having noticed that x^*x has the inverse in A_0, we define

2) Let a, b be positive elements in A and let $\lambda>0$. For $\lambda_1>0$ and $\lambda_2>0$ such as $\lambda=\lambda_1+\lambda_2$, $a+\lambda_1 e$ and $b+\lambda_2 e$ are invertible positive elements, and hence by Lemma 1, there are invertible positive elements c, d such that $a+\lambda_1 e=c^2$ and $b+\lambda_2 e=d^2$. Then

$$a+b+\lambda e=c^2+d^2=c^2(e+c^{-2}d^2).$$

Put $x=c^{-1}d$. Then x is invertible and

$$\sigma(x^*x)=\sigma(dc^{-1}\cdot c^{-1}d)=\sigma(c^{-2}d^2).$$

Therefore, $a+b+\lambda e$ is invertible if and only if $x^*x\geq 0$.

$$\Omega_1=\{\varphi \in \Omega \mid (x^*x)^\wedge(\varphi)>0\};$$
$$\Omega_2=\{\varphi \in \Omega \mid (x^*x)^\wedge(\varphi)<0\}.$$

Then $\Omega=\Omega_1 \cup \Omega_2$ and further Ω_1, Ω_2 are open and closed in Ω. Let

$$p=-\frac{1}{2}((x^*x)^{-1}a-e),$$

a being a positive element in A_0 such that $a^2=(x^*x)^2$ (use Lemma 1). Then it is readily verified that p is a projection in A_0. Since $\varphi^2((x^*x)^{-1}a)=\varphi((x^*x)^{-2}a^2)=\varphi(e)=1$,

$$\varphi((x^*x)^{-1}a)=\varphi((x^*x)^{-1})\varphi(a)=\begin{cases} 1 \text{ on } \Omega_1 \\ -1 \text{ on } \Omega_2. \end{cases}$$

Thus $\varphi(p)=0$ on Ω_1 and $\varphi(p)=1$ on Ω_2, that is to say, $\hat{p}$ is the characteristic function of Ω_2. Here $(e-p)x^*x(e-p)\geq 0$ and $px^*xp\leq 0$. Therefore, the proof will be completed by showing $p=0$. Suppose $p\neq 0$, i.e., Ω_2 is not empty. Then there is a constant $\beta<0$ such that $px^*xp\leq \beta p$, and hence we may assume without loss of generality that

$$px^*xp\leq -q$$

holds. Since $p(x^*x)^{-1}p$ is strictly negative in pA_0p, we may apply Lemma 1 to it in pA_0p, and so there is an element $h>0$ in $pA_0p\subset A_0$ such that $p(x^*x)^{-1}p=-h^2$. Let $y=pxh$. Then $py=yp=y$ and

$$y^*y\leq -p.$$

In fact, putting $z=xh-pxh$, we have $z^2=0$, and then by Lemma 2, $z^*z=-p-y^*y\geq 0$. Obviously the set B of all elements commuting with p is a closed $*$-subalgebra of A, and p belongs to the center of B. Keeping in mind that pBp is a C-symmetric Banach $*$-algebra with unit p and $y\in B$, and restricting our consideration to pBp, it is possible to assume that

$$y^*y\leq -e.$$

In this case, $(y^*y)^{-1}$ exists in $A(y^*y)$ and it is strictly negative. Applying Lemma 1 to $-(y^*y)^{-1}$, there is a positive element k in $A(y^*y)$ such that $(y^*y)^{-1}=-k^2$. Let $u=yk$. Then we have

$$u^*u=-e \text{ and } uu^*=-q,$$

where q is a projection. Put $q'=e-q$ and $v=uq'$, observe that $q'u=0$. Then $q'v=0, vq'=v$ and $v^2=0$. By Lemma 2, $v^*v\geq 0$. But $v^*v=q'u^*uq'=-q'$, which is a contradiction unless $q'=0$. Thus $q=e$ and so u is normal. Since A is C-symmetric, $u^*u=-e$ is impossible.

Now we are in position to prove the theorem. For B^*-algebras, the process of proving the symmetry after having established that the sum of positive elements is also positive is due to I. Kaplansky. This algebraic argument seems to be not immediately available for our case. Here we will present a different treatment for the symmetry.

Proof of Theorem. First we observe that for any element x in A,

$$r(x^*x)=\sup\{\lambda \mid \lambda \in \sigma(x^*x)\},$$

$r(\ ,\ )$ being the spectral radius. To prove this we may consider only the case when $\sigma(x^*x)$ contains a negative scalar. If this equality does not hold, then there is $\lambda_0>0$ such that $x^*x+\lambda_0 e$ is not invertible and $\lambda_0>\alpha=\sup\{\lambda\,|\,\lambda\in\sigma(x^*x)\}$. Since the non-zero portions of $\sigma(x^*x)$ and $\sigma(xx^*)$ are equal, $xx^*\leq\max(\alpha,0)e<\lambda_0 e$. Thus, if $x=h+ik$ with self-adjoint elements h and k, then, by Lemma 3,

$$x^*x+\lambda_0 e>x^*x+xx^*=h^2+k^2\geq 0,$$

which is a contradiction.

Suppose that $-1\in\sigma(x^*x)$. Then, by what we have observed, $\alpha\geq 1$. Now let us split the spectrum Ω of $A_0=A(x^*x)$ into three parts Ω_1, Ω_0 and Ω_2, where

$$\Omega_1=\{\varphi\in\Omega\,|\,(x^*x)^\wedge(\varphi)\leq -1/2\},$$
$$\Omega_2=\{\varphi\in\Omega\,|\,(x^*x)^\wedge(\varphi)\geq 1/2\}$$

and Ω_0 is the complement of $\Omega_1\cup\Omega_2$, and define the continuous function f_0 on $\Omega_1\cup\Omega_2$ by

$$f_0(\varphi)=\begin{cases}\sqrt{6}/2 & \text{on } \Omega_1\\ 1/2\sqrt{\alpha} & \text{on } \Omega_2.\end{cases}$$

Then, as is well known, there exists a real-valued continuous function f on Ω such that $\|f\|=\|f_0\|=\sqrt{6}/2$. Since every real valued function in $C(\Omega)$ can be approximated uniformly by real-valued functions in $\hat{A}_0$, we can select a self-adjoint element k in A_0 so that $\|f^2-\hat{k}^2\|<1/4\alpha$. Then, for $\varphi\in\Omega_2$, we have

$$(kx^*xk)^\wedge(\varphi)=\hat{k}^2(\varphi)(x^*x)^\wedge(\varphi)<1/2\alpha\cdot\alpha=1/2,$$

and for $\varphi\in\Omega_1\cup\Omega_0$,

$$(kx^*xk)^\wedge(\varphi)<(1+6\alpha)/4\alpha\cdot 1/2\leq 7/8.$$

Therefore, as seen above, $r(kx^*xk)<1$. However, for $\varphi_0\in\Omega$ such as $(x^*x)^\wedge(\varphi_0)=-1$,

$$(kx^*xk)^\wedge(\varphi_0)=-\hat{k}^2(\varphi_0)<(1-6\alpha)/4\alpha\leq -5/4<-1.$$

This contradicts $r(kx^*xk)<1$. That is, $-1\notin\sigma(x^*x)$. Consequently, we can conclude that x^*x+e is invertible for every element x in A.

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