61. *On the Evolution Equations with Finite Propagation Speed*

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1. Introduction. Let

$$(1.1)\qquad \left(\frac{\partial}{\partial t}\right)^m u(x,t)=\sum_{j<m} a_{\nu j}(x,t)\left(\frac{\partial}{\partial x}\right)^\nu\left(\frac{\partial}{\partial t}\right)^j u(x,t)$$

be an evolution equation defined on $(x,t)\in \boldsymbol{R}^l\times[0,T]\equiv\Omega$. We suppose all the coefficients are infinitely differentiable, and that for any time $t_0\in[0,T)$ and any initial data

$$\left(\frac{\partial}{\partial t}\right)^j u(x,t_0)=\varphi_j(x)\in\mathcal{D}\ (j=0,1,\cdots,m-1),$$

there exists a unique solution $u(x,t)$ for $t\in[t_0,T]$ in some functional space, say in $\mathcal{B}$ or in $\mathcal{D}_{L^p}$ $(1<p<+\infty)$.[1]

We say that (1.1) has a *finite propagation speed* if for any compact K in $\boldsymbol{R}^l$, there exists a finite $\lambda(K)$ (propagation speed) such that for any initial data $\Psi(x)\equiv(\varphi_0(x),\cdots,\varphi_{m-1}(x))\in\mathcal{D}$, with initial time t_0, whose support is contained in K, the support of the solution $u(x,t)$ is contained in

$$\bigcup_{\xi\in\mathrm{sup}[\Psi]}(\xi,t_0)+C^+_{\lambda(K)},$$

where $C^+_{\lambda(K)}$ is the cone defined by $\{(x,t)\,;\,|x|\le\lambda(K)t,\ t\ge 0\}$.

We say that (1.1) is a *kowalevskian in Ω*, if the coefficients $a_{\nu j}(x,t)$ appearing in the second member are identically zero if $|\nu|+j>m$. Our result is the

Theorem. *In order that* (1.1) *have a finite propagation speed, it is necessary that* (1.1) *be kowalevskian in Ω.*

This theorem was proved by Gårding [1] in the case where all the coefficients are constant. Now we can prove this theorem by the same method as in [2]. The detailed proof will be given in a forthcoming paper. In this Note, to make clear our reasoning, we argue on a simple equation.

2. Localizations of equation. Let

$$(2.1)\qquad \frac{\partial}{\partial t}u(x,t)=\sum_{|\nu|\le p} a_\nu(x,t)\left(\frac{\partial}{\partial x}\right)^\nu u(x,t)\equiv a_p\left(x,t\,;\frac{\partial}{\partial x}\right)u(x,t)$$

be an evolution equation, *not kowalevskian*, in Ω. Without loss of generality, we may assume that at the origin the second member of (2.1) is effectively of order $p(>1)$. We can find then a complex num-

1) With regards to these notations, see [2]. As the proof given later shows, this conditions can be replaced by weaker conditions.

ber $\zeta_0=\xi_0+i\eta_0$ ($\xi_0, \eta_0\neq 0$), such that

$$\mathrm{Re}\sum_{|\nu|=p} a_\nu(0,0)\zeta_0^\nu=2\delta>0. \tag{2.2}$$

Now take a function $\beta(x)\in\mathcal{D}$ of small support taking the value 1 in a neighborhood of $x=0$. Apply $\beta(x)$ to (2.1), then

$$\frac{\partial}{\partial t}(\beta u)=a_p\left(x,t;\frac{\partial}{\partial x}\right)(\beta u)+\sum_\mu a_{p,\mu}\left(x,t;\frac{\partial}{\partial x}\right)(\beta^{(\mu)}u), \tag{2.3}$$

where the coefficients may be supposed, by changing these outside the support of $\beta(x)$, to be near the values at the origin (localization in the x-space) if we restrict the variable t to a small neighborhood of zero, say $t\leq\varepsilon$. Here the order of $a_{p,\mu}$ is equal to $(p-|\mu|)$.

Now by the hypothesis of the well-posedness of (2.1), there exists a constant C and h independent of (x_0, t_0) such that it hold for any initial data $u(x,0)\in\mathcal{D}$,

$$\begin{aligned}|u(x_0,t_0)|&\leq C\sum_{|\alpha|\leq h}\sup_{x\in\mathbf{R}^l}|D^\alpha u(x,0)|,\ \text{or}\\ &\leq C\sum_{|\alpha|\leq h}\|D^\alpha u(x,0)\|_{L^p(\mathbf{R}^l)},\end{aligned} \tag{2.4}$$

for any $x_0\in\mathrm{supp}[\beta]$ and $t_0\in[0,T]$. So, let us denote by $T_y(x_0,t_0)$ the distribution (in y) defined by

$$u(x_0,t_0)=\langle T_y(x_0,t_0), u(y,0)\rangle. \tag{2.5}$$

Let us suppose that (2.1) *has a finite propagation speed.* This implies that there exists a positive constant λ such that for $x_0\in\mathrm{supp}[\beta]$, and $t_0\in[0,\varepsilon]$, (ε small),

$$\mathrm{supp}[T_y(x_0,t_0)]\subset B_{\lambda t_0}(x_0)\equiv\{y\,;\,|y-x_0|\leq\lambda t_0\}. \tag{2.6}$$

Now in any case of (2.4), it is shown that we can sharpen the inequality (2.4) in the following way:

$$|\langle T_y(x_0,t_0),u(y,0)\rangle|\leq C'\sum_{|\alpha|\leq h}\sup_{|y-x_0|\leq\lambda t_0}|D^\alpha u(y,0)|, \tag{2.7}$$

where C' depends on C, h and l, but does not depend on (x_0,t_0).

Let $\hat{u}_0(\eta)$ be a continuous function $\not\equiv 0$ whose support is contained in a unit sphere with center at the origin, and let $u_0(x)$ be the inverse Fourier image. We define a sequence of solutions $u_n(x,t)$ of (2.1) by the initial data,

$$u_n(x,0)=\gamma(x)e^{nx\cdot\zeta_0}u_0(x)\equiv\gamma(x)e^{nx\cdot\xi_0}e^{inx\cdot\eta_0}u_0(x)\in\mathcal{D},$$

where $\gamma(x)$ is a function of $\mathcal{D}$ which takes the value 1 on the set $|x|\leq L$ (sufficiently large).

Next apply $e^{-nx\cdot\xi_0}$ to (2.3) after replacing u by u_n, it becomes

$$\begin{aligned}\frac{\partial}{\partial t}(\beta e^{-nx\cdot\xi_0}u_n)&=a_p\left(x,t;\frac{\partial}{\partial x}+n\xi_0\right)(\beta e^{-nx\cdot\xi_0}u_n)\\ &\quad+\sum a_{p,\mu}\left(x,t;\frac{\partial}{\partial x}+n\xi_0\right)(\beta^{(\mu)}e^{-nx\cdot\xi_0}u_n).\end{aligned} \tag{2.8}$$

Now let us estimate the function

$$v_n(x,t)=e^{-nx\cdot\xi_0}u_n(x,t). \tag{2.9}$$

By (2.7),

$$
\begin{aligned}
|e^{-nx\cdot\xi_0}u_n(x,t)| &= |\langle T_y(x,t),\ e^{-nx\cdot\xi_0}\gamma(y)e^{iny\cdot\zeta_0}u_0(y)\rangle| \\
&= |\langle T_y(x,t),\ e^{n(y-x)\cdot\xi_0}e^{iny\cdot\eta_0}u_0(y)\rangle| \\
&\leq C' \sum_{|\alpha|\leq h} \sup_{|y-x|\leq\lambda t} |D^\alpha\{e^{n(y-x)\cdot\xi_0}e^{iny\cdot\eta_0}u_0(y)\}|.
\end{aligned}
$$

So we have

(2.10) $\quad |v_n(x,t)| \leq C'' n^h \exp(n\lambda|\xi_0|t), \quad \text{for} \quad x\in \text{supp}[\beta],$

and $\quad t\in[0,\varepsilon],$

where C'' is a constant independent of (x, t) and n. Remarking this, let $\alpha(\eta)$ be a function of $\mathcal{D}$ having its support in a small neighborhood of η_0, and taking the value 1 in a neighborhood of η_0. Finally, putting

(2.11) $$\alpha_n(\eta)=\alpha(\eta/n),$$

we define the convolution operator $\alpha_n(D)$. Applying this to (2.8), we get a new equation localized in both x and η spaces:

(2.12)
$$
\begin{aligned}
\frac{\partial}{\partial t}(\alpha_n(D)\beta v_n) &= a_p\Big(x,t;\frac{\partial}{\partial x}+n\xi_0\Big)(\alpha_n(D)\beta v_n) \\
&\quad + \sum a_{p,\mu}\Big(x,t;\frac{\partial}{\partial x}+n\xi_0\Big)(\alpha_n(D)\beta^{(\mu)}v_n) \\
&\quad + [\alpha_n(D), a_p](\beta v_n) + \sum [\alpha_n(D), a_{p,\mu}](\beta^{(\mu)}v_n).
\end{aligned}
$$

3. Energy inequality. Let us consider the following equation:

(3.1) $$\frac{\partial}{\partial t}(\alpha_n(D)w(x,t)) = a_p\Big(x,t;\frac{\partial}{\partial x}+n\xi_0\Big)(\alpha_n(D)w)+f(x,t).$$

Taking account of (2.2), it is shown that the following inequality holds for $t\in[0,\varepsilon]$:

(3.2) $$\frac{d}{dt}\|\alpha_n(D)w(x,t)\| \geq \delta n^p\|\alpha_n(D)w(x,t)\| - \|f(x,t)\|,$$

where $\|\cdot\|$ denotes the L^2-norm in $\boldsymbol{R}^l$. In fact, on the support of $\alpha_n(\eta)$, the symbol of $a_p\Big(x,t;\dfrac{\partial}{\partial x}+n\xi_0\Big)$ behaves like $a_p(x,t;n\zeta_0)$. Now, in view of (2.11), we have

$$|\alpha_n^{(\kappa)}(\eta)| \leq \text{constant. } n^{-|\kappa|}.$$

So, if we develop the commutator $[\alpha_n(D), a_p]$, it holds:

$$[\alpha_n, a_p] = \sum_{|\kappa|=1}^{m} i^{|\kappa|}\partial_x^\kappa a_p\Big(x,t;\frac{\partial}{\partial x}+n\xi_0\Big)\alpha_n^{(\kappa)}(D) + R_{m,p},$$

where $\qquad \|R_{m,p}(u)\| \leq \text{constant. } n^{l+p-m-1}\|u\|,$

where, let us recall, l is the dimension of the space and p is the order of a_p. The same kind of inequalities holds for $[\alpha_n, a_{p,\mu}]$. So, if we take

(3.3) $$m=h+l,$$

we shall have, in view of (2.10), (2.12) and (3.2):

$$
\begin{aligned}
\frac{d}{dt}\|\alpha_n\beta v_n\| &\geq \delta n^p\|\alpha_n\beta v_n\| - c\,n^p\sum_{1\leq|\kappa|\leq m}\|\alpha_n^{(\kappa)}\beta v_n\| \\
&\quad - c\,n^{p-1}\sum_{1\leq|\kappa|\leq m-1,\,|\mu|=1}\|\alpha_n^{(\kappa)}\beta^{(\mu)}v_n\| - \cdots - c\,n^{p-i}\sum_{1\leq|\kappa|\leq m-i,\,|\mu|=i}\|\alpha_n^{(\kappa)}\beta^{(\mu)}v_n\|
\end{aligned}
$$

$$\cdots - c \sum_{|\mu|=p} \|\alpha_n \beta^{(\mu)} v_n\| - c\, n^{p-1} \exp(n|\xi_0|\lambda t).$$

Namely

$$(3.4) \quad \frac{d}{dt}\|\alpha_n \beta v_n\| \geq \delta n^p \|\alpha_n \beta v_n\| - c\, n^p \sum_{1 \leq |\kappa|+|\mu| \leq m} \|\alpha_n^{(\kappa)} n^{-|\mu|} \beta^{(\mu)} v_n\| - c\, n^{p-1} \exp(n|\xi_0|\lambda t).$$

Define

$$S_n(t) = \sum_{|\kappa|+|\mu| \leq m} C_0^{|\kappa|+|\mu|} \|\alpha_n^{(\kappa)} n^{-|\mu|} \beta^{(\mu)} v_n\|.$$

This means that we consider all the functions $\alpha_n^{(\kappa)} \beta^{(\mu)} v_n$ instead of $\alpha_n \beta v_n$ in (2.12). Then we shall have the same kinds of inequalities as (3.4). So, if we choose C_0 large enough, summing up all the inequalities thus obtained, we shall have

$$S_n'(t) \geq \frac{\delta}{2} n^p S_n(t) - c'\, n^{p-1} \exp(n|\xi_0|\lambda t).$$

Hence

$$S_n(t) \geq S_n(0) \exp\left(\frac{\delta}{2} n^p t\right) - c'\, n^{p-1} \exp\left(\frac{\delta}{2} n^p t\right) \int_0^t \exp\left(-\frac{\delta}{2} n^p \tau\right) \exp(n|\xi_0|\lambda \tau) d\tau.$$

Taking account of $\|\alpha_n(D)\beta(x)v_n(x, 0)\| = \|\alpha_n(D)\beta(x)e^{inx\cdot\eta_0}u_0(x)\|$, and in view of [2], we see that $\|\alpha_n \beta v_n(x, 0)\| \geq \delta_0 \,(>0)$ for n large. A fortiori, it holds $S_n(0) \geq \delta_0$ for n large. Thus,

$$(3.5) \quad S_n(t) \geq \frac{\delta_0}{2} \exp\left(\frac{\delta}{2} n^p t\right) \quad \text{for} \quad t \in [0, \varepsilon], \quad n \text{ large}.$$

In fact, for n large, since $p>1$, we have $n|\xi_0|\lambda < \frac{\delta}{4} n^p$. Then

$$\int_0^t \exp\left(-\frac{\delta}{2} n^p \tau\right) \exp(n|\xi_0|\lambda \tau) d\tau \leq \int_0^t \exp\left(-\frac{\delta}{4} n^p \tau\right) d\tau \leq \frac{1}{n^p} \int_0^\infty \exp\left(-\frac{\delta}{4} \tau\right) d\tau.$$

On the other hand, (2.10) shows that $S_n(t) \leq \text{const.}\ n^h \exp(n|\xi_0|\lambda t)$. This inequality is not compatible with (3.5) unless $t=0$. Thus we proved the Theorem in the Introduction by contradiction.

References

[1] L. Gårding: Linear hyperbolic partial differential equations with constant coefficients. Acta Math., **85**, 1–62 (1951).

[2] S. Mizohata: Some remarks on the Cauchy problem. J. of Math. Kyoto Univ., **1**, 109–127 (1961).