76. On a Certain Type of Differential Hopf Algebras*)

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In [1] we have introduced strange differential Hopf structures arising from K-theory and have called them differential near Hopf algebras. One of the purposes of this paper is to find a general theory in order to make these strange differential Hopf algebras fit with the usual differential Hopf algebras.

Our main result is a generalization of a criterion of coprimitivity of Hopf algebras [5]. This enables us to use biprimitive form spectral sequences due to Browder [3] in researches of K-theory of H-spaces.

The detailed proofs will be published elsewhere.

1. By a G_2-module $M=M_0\oplus M_1$ we mean a Z_2-graded module over a field K. M has a canonical involution σ such that

$$\sigma\,|\,M_0=1 \quad \text{and} \quad \sigma\,|\,M_1=-1.$$

All algebraic structures such as algebras, coalgebras, differential algebras, etc., will be understood those over certain underlying G_2-modules [1]. In the present work, all algebras (or coalgebras) are equipped with augmentations and units (or counits), but are not necessarily associative.

Let M and N be differential G_2-modules. $M\otimes N$ is also a differential G_2-module. The usual switching map

$$T: M\otimes N \rightarrow N\otimes M$$

is an isomorphism of differential G_2-modules. Pick $\lambda\in K$. We define the *λ-modified switching* map

$$T_\lambda: M\otimes N \rightarrow N\otimes M$$

by $T_\lambda=(1+\lambda\cdot d\sigma\otimes d)T$. T_λ is also an isomorphism of differential G_2-modules and involutive, i.e., $T_\lambda^2=1$.

Generalizing the above T_λ, we can define the *λ-modified permutations* of tensor factors so that $\mathfrak{S}_n$ acts as a group of automorphisms of the differential G_2-module $M^{\otimes n}=M\otimes\cdots\otimes M$.

Our first basic idea is to replace the switching maps and permutations of tensor factors in the ordinary theory of Hopf algebras [5] by λ-modified ones and to construct a theory suitable to Hopf structures derived from mod p K-theory.

2. Let A and B be differential algebras (or coalgebras). Putting

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$$\varphi_\lambda=(\varphi\otimes\varphi)(1\otimes T_\lambda\otimes 1): A\otimes B\otimes A\otimes B\to A\otimes B$$

(or

$$\psi_\lambda=(1\otimes T_\lambda\otimes 1)(\psi\otimes\psi): A\otimes B\to A\otimes B\otimes A\otimes B)$$

and defining the augmentation, unit (or counit) and differential as usual, $A\otimes B$ becomes a differential algebra (or coalgebra), which has a multiplication φ_λ (or comultiplication ψ_λ) different from that of the ordinary tensor product. We call this the *λ-modified tensor product* of A and B and denote it by $(A\otimes B)_\lambda$. Thus $(A\otimes B)_0=A\otimes B$ is the ordinary tensor product.

If a differential algebra (or coalgebra) A satisfies the relation

$$\varphi T_\lambda=\varphi \qquad (\text{or} \quad T_\lambda\psi=\psi),$$

then we call A is *λ-commutative.*

3. Let A be an algebra (or coalgebra). We generalize Browder's filtration [1, 3] to non-associative cases and obtain a decreasing filtration $\{F^kA,\ k\geqq 0\}$ of the algebra A (or an increasing filtration $\{G^kA,\ k\geqq 0\}$ of the coalgebra A). The associated graded G_2-module is denoted by

$$E_0(A)=\sum_{k\geqq 0} E_0^kA, \quad E_0^kA=F^kA/F^{k+1}A,$$

for an algebra A, and by

$${}_0E(A)=\sum_{k\geqq 0} {}_0E^kA, \quad {}_0E^kA=G^kA/G^{k-1}A,$$

for a coalgebra A. The usual basic properties of these filtrations [1, 3] are retained.

If an algebra (or coalgebra) A satisfies the following condition

(3.1) $\bigcap_{k\geqq 0} F^kA=\{0\}$ (or $\bigcup_{k\geqq 0} G^kA=A$)

we call A *semi-connected.* Remark that a graded connected algebra (or coalgebra) is semi-connected.

If A is semi-connected and of finite dimension, then $E_0(A)$(or ${}_0E(A)$) is isomorphic to A as a G_2-module.

Usually a decreasing filtration topologizes A. For an algebra A we topologize A by an F-filtration. Then A is a Hausdorff space if it is semi-connected.

(3.2) *Let A be a semi-connected algebra* (*or coalgebra*). *Then* $\bar{A}=\{0\}$ *if and only if* $Q(A)=\{0\}$ (*or* $P(A)=\{0\}$).

Let $f: A\to B$ be a morphism of algebras. If $f(A)$ is dense in B (topologized by the F-filtration) then we call f *almost surjective.*

(3.3) *Let $f: A\to B$ be a morphism of algebras. $f: A\to B$ is almost surjective if and only if $Q(f): Q(A)\to Q(B)$ is surjective.*

As a dual to the above proposition we obtain

(3.3*) *Let $f: A\to B$ be a morphism of coalgebras and assume A to be semi-connected. $f: A\to B$ is injective if and only if $P(f)$: $P(A)\to P(B)$ is injective.*

4. Let A be a (differential) algebra as well as a (differential) coalgebra. If the unit and the augmentation of the algebra coincide

with the augmentation and the counit of the coalgebra, then A is called a (differential) *quasi pre* Hopf algebra. Furthermore, if it is associative as an algebra as well as a coalgebra, it is called a (differential) *pre* Hopf algebra [1].

If A is a differential quasi pre Hopf algebra, then we can discuss the F- and G-filtration of A. Both filtrations are d-stable [1] and determine the spectral sequences

$$E_r(A)=\sum_{n\geqq 0} E_r^n A \quad \text{and} \quad {}_rE(A)=\sum_{n\geqq 0} {}_rE^n A,$$

$r\geqq 0$, as usual. These are spectral sequences of algebras and coalgebras, respectively.

If a differential (quasi) pre Hopf algebra A satisfies

(4.1) $\psi\varphi=(\varphi\otimes\varphi)(1\otimes T_\lambda\otimes 1)(\psi\otimes\psi)$

for some $\lambda\in K$, then we call A a *λ-modified differential (quasi) Hopf algebra*, or simply a *(quasi) (d, λ)-Hopf algebra*. Thus, to say that A is a quasi (d, λ)-Hopf algebra is equivalent to say that

$$\psi : \mathrm{A}\to(A\otimes A)_\lambda$$

is a morphism of differential algebras or that

$$\varphi : (A\otimes A)_\lambda\to A$$

is a morphism of differential coalgebras. Thus ψ and φ induces morphisms

$$E_r(\psi) : E_r(A)\to E_r((A\otimes A)_\lambda)$$

and

$${}_rE(\varphi) : {}_rE((A\otimes A)_\lambda)\to {}_rE(A)$$

of terms of spectral sequences for $r\geqq 0$. Since there hold Künneth relations for λ-modified tensor products in each term of both spectral sequences, $E_r(\psi)$ (or ${}_rE(\varphi)$) defines a comultiplication (or a multiplication) in $E_r(A)$ (or ${}_rE(A)$), and the latter becomes a graded connected quasi (d, λ)-Hopf algebra for $r=0$ and a graded connected quasi differential Hopf algebra for $r\geqq 1$.

(4.2) *$E_0(A)$ is primitive and ${}_0E(A)$ is coprimitive.* (Cf., [3].)

5. Let A be a differential algebra (or coalgebra) and $\lambda\in K$. Let $p=\mathrm{Char}\ K$ and we suppose $p\neq 0$. A λ-modified cyclic permutation $C_\lambda : (A^{\otimes p})_\lambda\to(A^{\otimes p})_\lambda$ is a morphism of differential algebras (or coalgebras). Put $\Delta_\lambda=1-C_\lambda$ and $\Sigma_\lambda=\sum_{i=0}^{p-1} C_\lambda^i$. Define

$$\Phi_\lambda A=\mathrm{Ker}\ \Sigma_\lambda/\mathrm{Im}\ \Delta_\lambda$$

and

$$\Psi_\lambda A=\mathrm{Ker}\ \Delta_\lambda/\mathrm{Im}\ \Sigma_\lambda.$$

Then we have

(5.1) i) *When A is a differential algebra, $\Psi_\lambda A$ is a differential algebra.* ii) *When A is a differential coalgebra, $\Phi_\lambda A$ is a differential coalgebra.*

Now let A be a (quasi) (d, λ)-Hopf algebra for $\lambda\in K$. $\Psi_\lambda A$ and $\Phi_\lambda A$ are differential algebra and coalgebra respectively. On the other hand we can prove that the canonical map $\Psi_\lambda A \to \Phi_\lambda A$ is an isomorphism. Identifying them by this canonical isomorphism we get

(5.2) *$\Phi_\lambda A = \Psi_\lambda A$ is a (quasi) Hopf algebra.*

We call $\Phi_\lambda A$ the *derived (quasi) Hopf algebra* of A.

6. Let A be a quasi (d, λ)-Hopf algebra. Assume that $p \neq 0$ and the multiplication (or the comultiplication) of A is associative and λ-commutative. We define a map

$$\xi'_\lambda : \text{Ker } \Sigma_\lambda \to A \quad (\text{or } \eta'_\lambda : A \to \text{Coker } \Sigma_\lambda)$$

by $\xi'_\lambda = \varphi_{p-1} i$ (or $\eta'_\lambda = \pi \psi_{p-1}$), where $\varphi_{p-1} = \varphi(\varphi \otimes 1) \cdots (\varphi \otimes 1 \otimes \cdots \otimes 1)$: $(A^{\otimes p})_\lambda \to A$, $\psi_{p-1} = (\psi \otimes 1 \otimes \cdots \otimes 1) \cdots (\psi \otimes 1)\psi : A \to (A^{\otimes p})_\lambda$, i: Ker $\Sigma_\lambda \to (A^{\otimes p})_\lambda$ is the inclusion and $\pi : (A^{\otimes p})_\lambda \to$ Coker Σ_λ is the projection. Since φ(or ψ) is λ-commutative we have

$$\varphi_{p-1} \Delta_\lambda = 0 \quad (\text{or } \Delta_\lambda \psi_{p-1} = 0).$$

Passing to quotient (or restricting range) we have the induced map

$$\xi_\lambda : \Phi_\lambda A \to A \quad (\text{or } \eta_\lambda : A \to \Psi_\lambda A).$$

Here we obtain

(6.1) *The above map ξ_λ (or η_λ) is a morphism of (d, λ)-Hopf algebras.*

Now we can state our main theorems.

Theorem 1. *Let $\lambda \in K$ and A be a quasi (d, λ)-Hopf algebra which is semi-connected as a coalgebra. If A is coprimitive then the multiplication is associative, λ-commutative and, when $p \neq 0$, $\bar{\xi}_\lambda$ = zero map.*

The proof is based on (3.3*). Dually we obtain

Theorem 2. *Let $\lambda \in K$ and A be a quasi (d, λ)-Hopf algebra which is semi-connected as an algebra. If A is primitive then the comultiplication is associative, λ-commutative and, when $p \neq 0$ and $\Psi_\lambda A$ is semi-connected as an algebra, $\bar{\eta}_\lambda$ = zero map.*

7. As inverses to the above Theorems we obtain the following

Theorem 3. *Let $\lambda \in K$ and A be a quasi (d, λ)-Hopf algebra which is semi-connected as an algebra. Suppose that p is odd or that $p = 2$ and $\lambda d = 0$. If the multiplication is associative, λ-commutative and $\bar{\xi}_\lambda$ = zero map, then A is coprimitive.*

Theorem 4. *Let $\lambda \in K$ and A be a quasi (d, λ)-Hopf algebra which is semi-connected as a coalgebra. Suppose that p is odd or that $p = 2$ and $\lambda d = 0$. If the comultiplication is associative, λ-commutative and $\bar{\eta}_\lambda$ = zero map, then A is primitive.*

In case $p = 2$ and $\lambda d \neq 0$ these theorems are not proved. Nevertheless this is not an obstruction to our applications. In fact,

Theorem 5. *The conclusions of Theorems* 1 *and* 2 *are hereditary to $H(A)$.*

If A is graded and connected, then A, $\Psi_\lambda A$ and $H(A)$ are semi-connected as algebras as well as coalgebras. Thus

Theorem 6. *Let $p \neq 0$ and A be a quasi (d, λ)-Hopf algebra. Then*

$E_r(A)$ is primitive and ${}_rE(A)$ is coprimitive for all $r\geqq 0$. If A is primitive or coprimitive then ${}_rE(A)$ or $E_r(A)$ is biprimitive for all $r\geqq 0$.

8. Let $p\neq 0$ and let A be a (d, λ)-Hopf algebra of finite dimension. If A is semi-connected as an algebra then ${}_0E(E_0(A))$ is a biprimitive (d, λ)-Hopf algebra which is isomorphic to A as a G_2-module. Thus it is a *biprimitive form* of A [3]. When A is semi-connected as a coalgebra $E_0({}_0E(A))$ is a biprimitive form of A. Thus, if A is semi-connected either as an algebra or as a coalgebra, the assumption of finite dimensionality allows us to discuss the "biprimitive form spectral sequence" due to Brower [3].

Let X be a connected H-space which has the homotopy type of a finite CW-complex. $K^*(X;Z_p)$ [2] is an example of quasi (d, λ)-Hopf algebras. Since X is finite dimensional the usual filtration of $K^*(X;Z_p)$, defined by skeletons, is multiplicative and tends to zero. This filtration is superior to our F-filtration, so $K^*(X;Z_p)$ is semi-connected as an algebra. The E_2-term is the d_1-homology of $E_1 = K^*(X;Z_p)$, and its F-filtration is majorated by the induced filtration which tends to zero. Thus the E_2-term is semi-connected as an algebra. Similarly, every term of the Bockstein spectral sequence is semi-connected as an algebra. Thus we obtain

Theorem 7. *Let X be a connected H-space which has the homotopy type of a finite CW-complex. We have a biprimitive form spectral sequence which starts from a biprimitive form of $K^*(X;Z_p)$ and ends at that of $(K^*(X)/\text{Torsions})\otimes Z_p$.*

This can be used to compute some $K^*(G)$.

Remark. $K^*(X;Z_p)$ is not necessarily semi-connected as a coalgebra. An example is $K^*(SO(n);Z_2)$.

References

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